Self-gravitating systems and Balescu-Lenard equation

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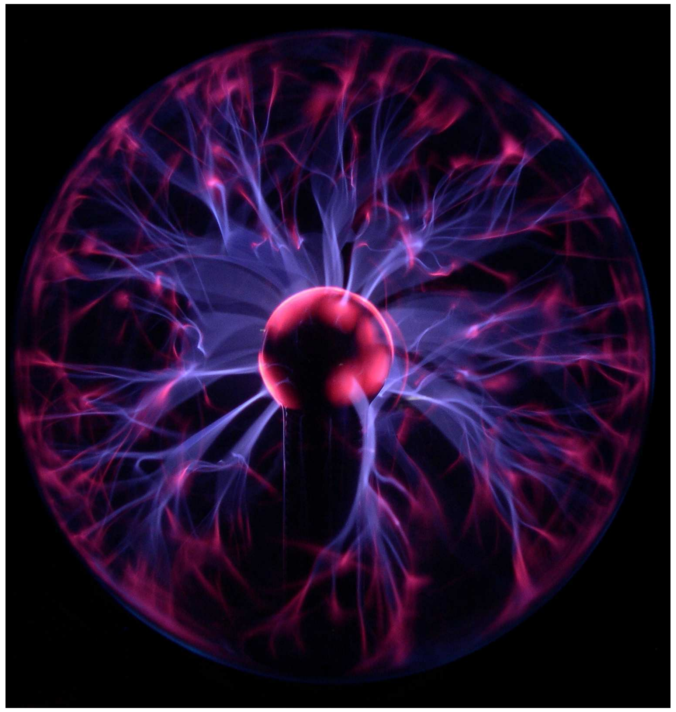
Long-term relaxation

How do systems **diffuse**?

Local
Brownian diffusion



Homogeneous
Plasma diffusion

Inhomogeneous
Galaxy diffusion

Fluctuation-Dissipation Theorem

Diffusion ⇄ **Noise**

Same process occur in galaxies, but:

Gravity is **long-range**
+ Stars follow **orbits** and **resonate**
+ Galaxies **amplify** perturbations

How do galaxies evolve on cosmic timescales?

The gravitational Balescu-Lenard equation

What does it require?

What is it?

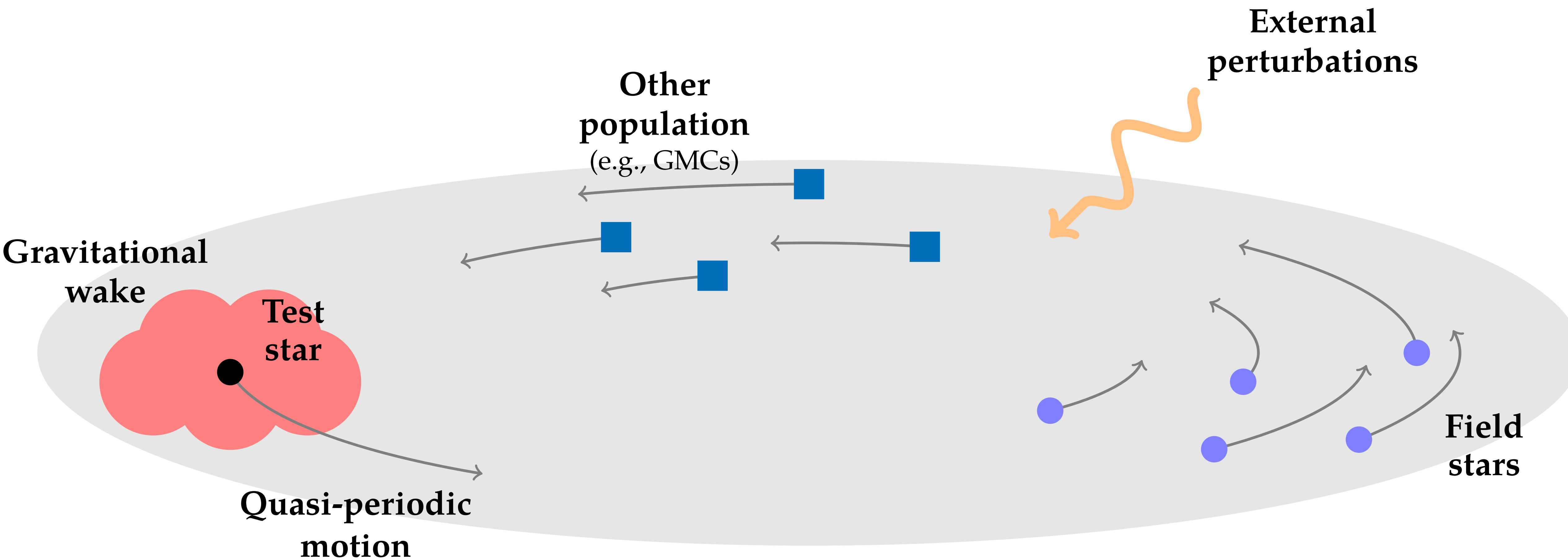
Where does it come from?

Does it work?

What's next?

What does the Balescu-Lenard Eq. require?

Galactic evolution on cosmic timescales



Galaxies are:

+ **Inhomogeneous** (complex trajectories)	*Angle-action coordinates*
+ **Relaxed** (equilibrium states)	*Quasi-stationary states*
+ **Resonant** (orbital frequencies)	*Fast timescale vs. cosmic timescale*
+ **Degenerate** (in some regions)	*Frequency commensurability*
+ **Self-gravitating** (amplification of perturbations)	*Linear response theory*
+ **Discrete** (finite-N effects) + **Perturbed** (effects of the environment)	*Nature vs. Nurture*

What does it require?

Inhomogeneous

$$(\mathbf{x}, \mathbf{v}) \downarrow (\boldsymbol{\theta}, \mathbf{J})$$

Angle-Action coordinates

Relaxed

$$F = F(\mathbf{J}, t)$$

Quasi-stationary states

Resonant

$$\Omega(\mathbf{J}) = \partial H_0 / \partial \mathbf{J}$$

Fast/Slow timescale

Self-gravitating

$$\frac{1}{|\,\varepsilon(\omega)\,|}$$

Linear response theory

Discrete & Perturbed

$$\frac{1}{N}$$

Finite-N effects

What does it require?

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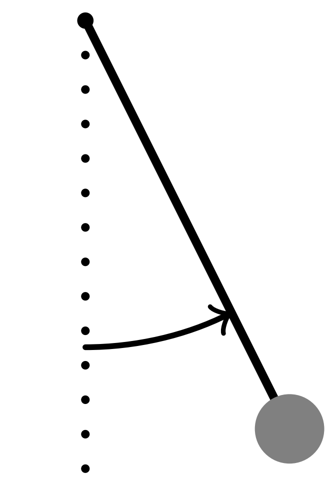
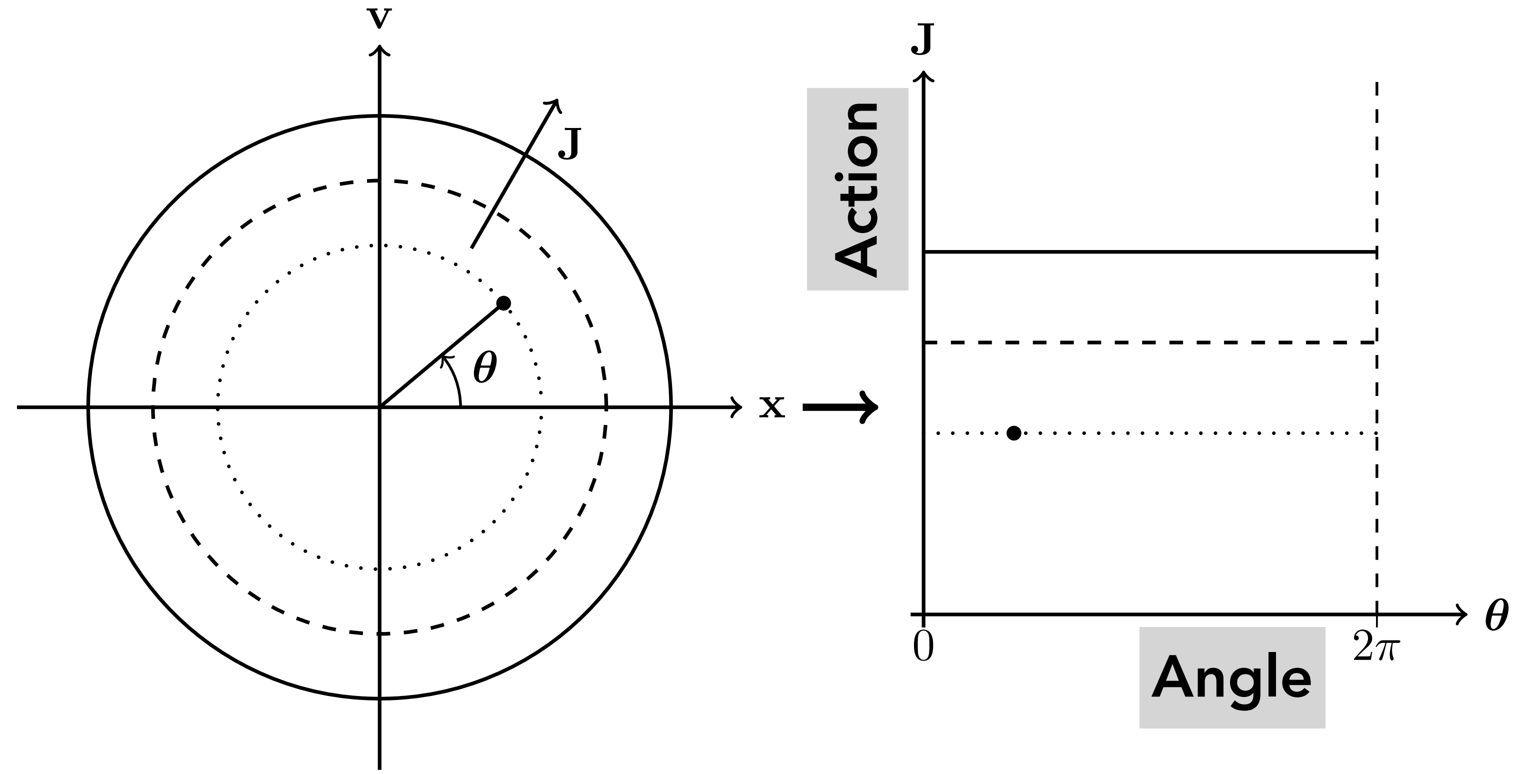
Finite-N effects

Inhomogeneous systems

+ **Label** orbits with **integrals of motion**



Pendulum



+ **Angle-Action coordinates**

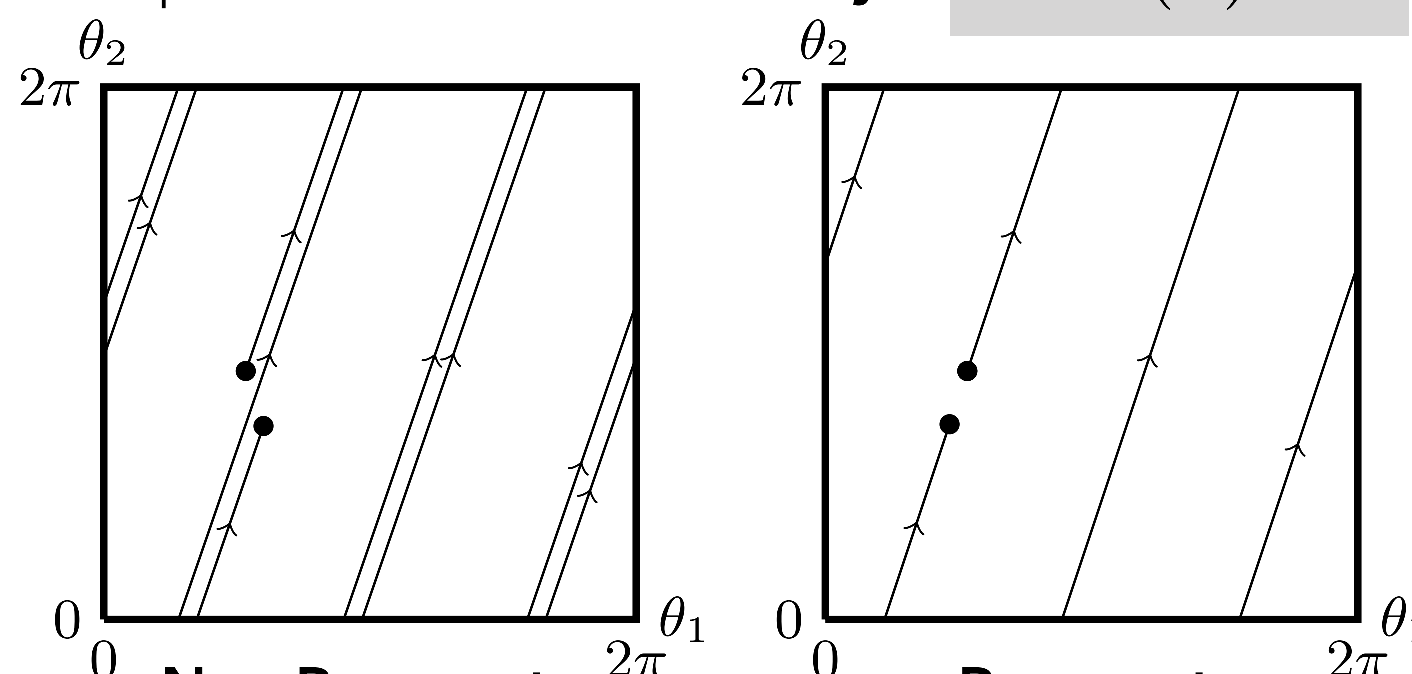
$$\begin{cases} \boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + t\,\boldsymbol{\Omega}(\mathbf{J}) \\ \mathbf{J}(t) = \mathrm{cst.} \end{cases}$$

Trajectories become **straight lines**

+ **Relaxation**

$$\xrightarrow{(\mathrm{few})\, t_{\mathrm{cross}}} \quad F = F(\mathbf{J}, t)$$

+ Frequencies' **commensurability** : $\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) = 0$

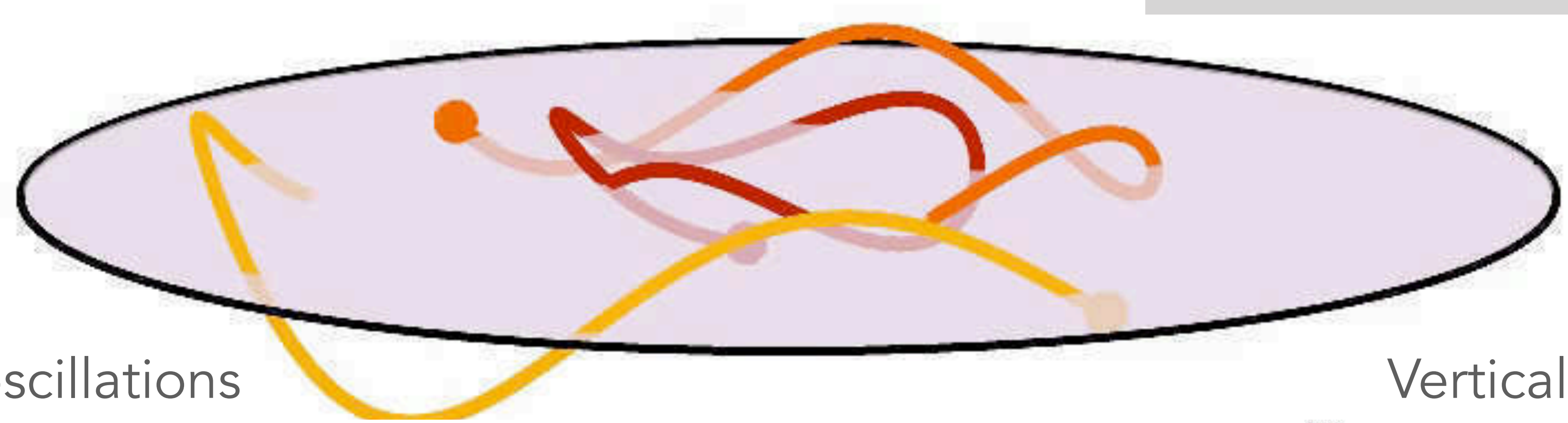


Non-Resonant **Resonant**

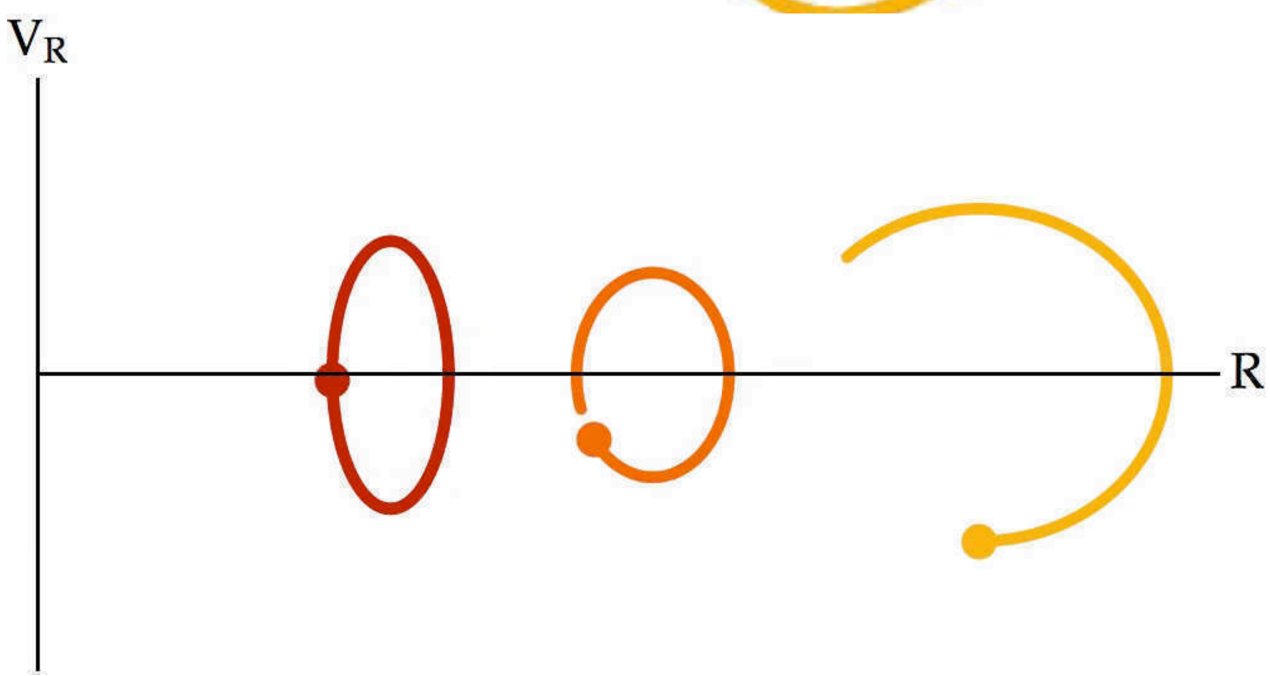
Example: Orbits in a disc

Integrable orbits $\Phi_0 = \Phi_0(R, z)$

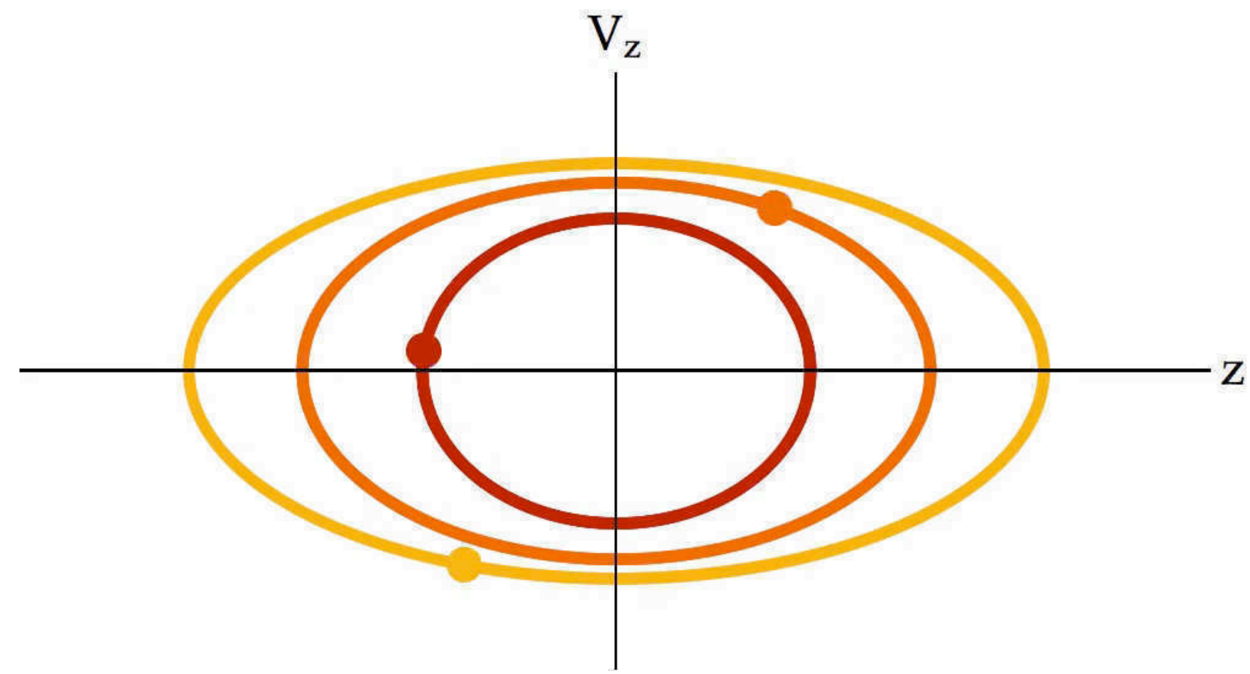
$$\begin{cases} \boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + t\,\boldsymbol{\Omega}(\mathbf{J}) \\ \mathbf{J}(t) = \text{cst.} \end{cases}$$



Radial oscillations



Vertical oscillations



Actions $\mathbf{J} = \left(J_\phi\,, J_r\,, J_z\right)$

Frequencies $\boldsymbol{\Omega} = \left(\Omega_\phi, \Omega_r, \Omega_z\right)$

What does it require?

Inhomogeneous

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Angle-Action coordinates

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Resonant

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Fast/Slow timescale

Self-gravitating

$$\frac{1}{|\varepsilon(\omega)|}$$

Linear response theory

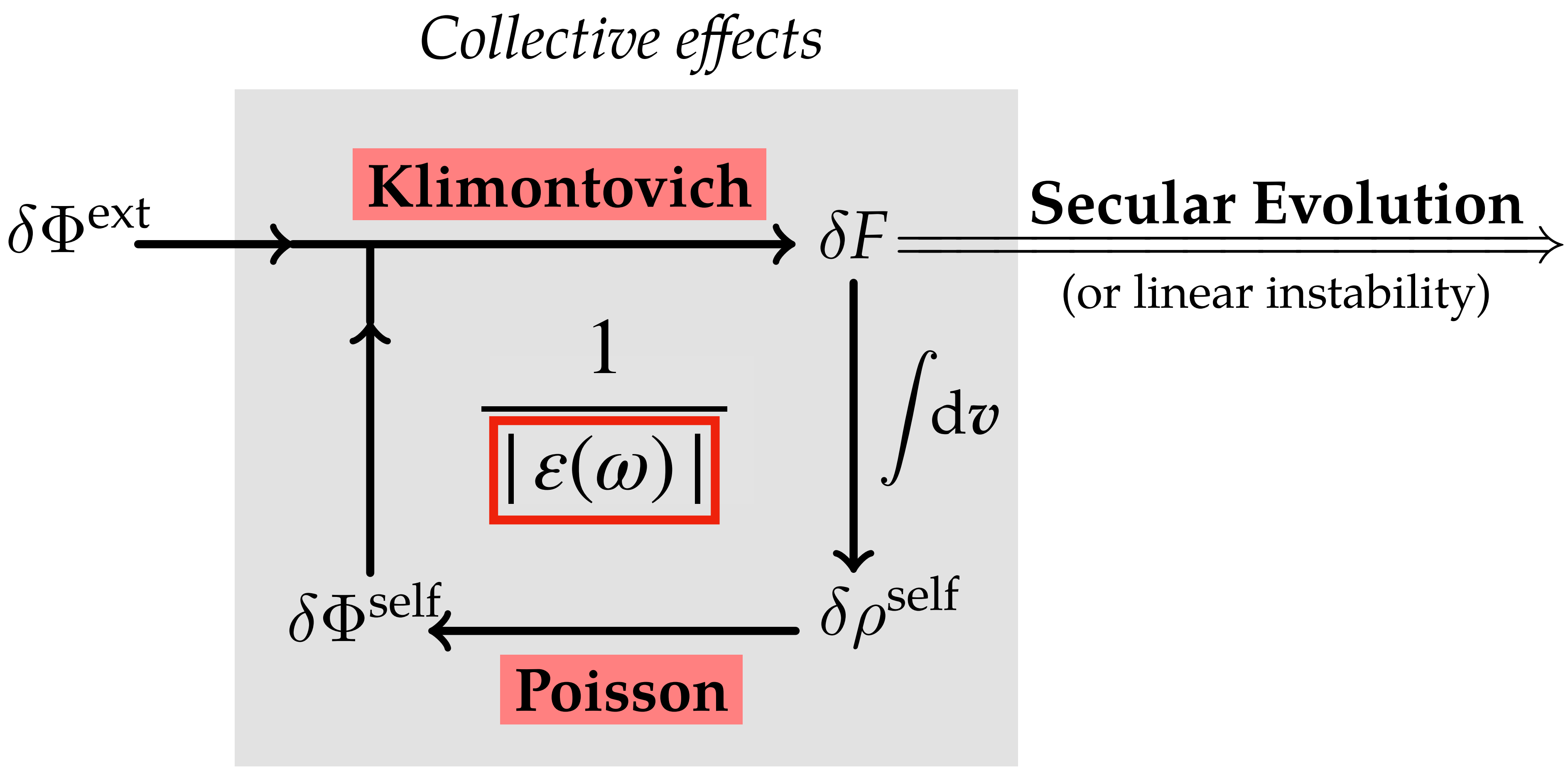
Discrete & Perturbed

$$\frac{1}{N}$$

Finite-N effects

Collective effects

Self-gravitating amplification

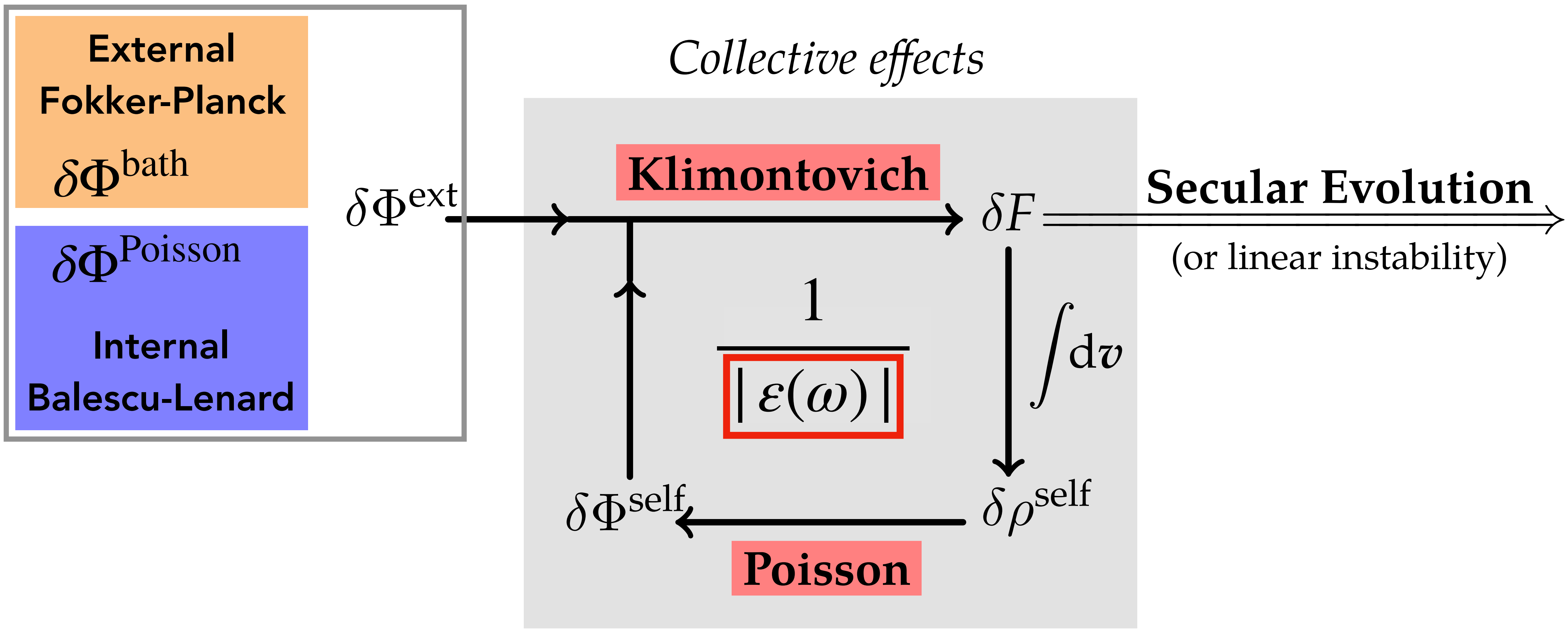


Gravitational polarisation essential to

+ *Cause dynamical instabilities*
+ *Induce* ***dynamical friction*** *and* ***mass segregation***
+ ***Accelerate/Slow down*** *secular evolution*

Collective effects

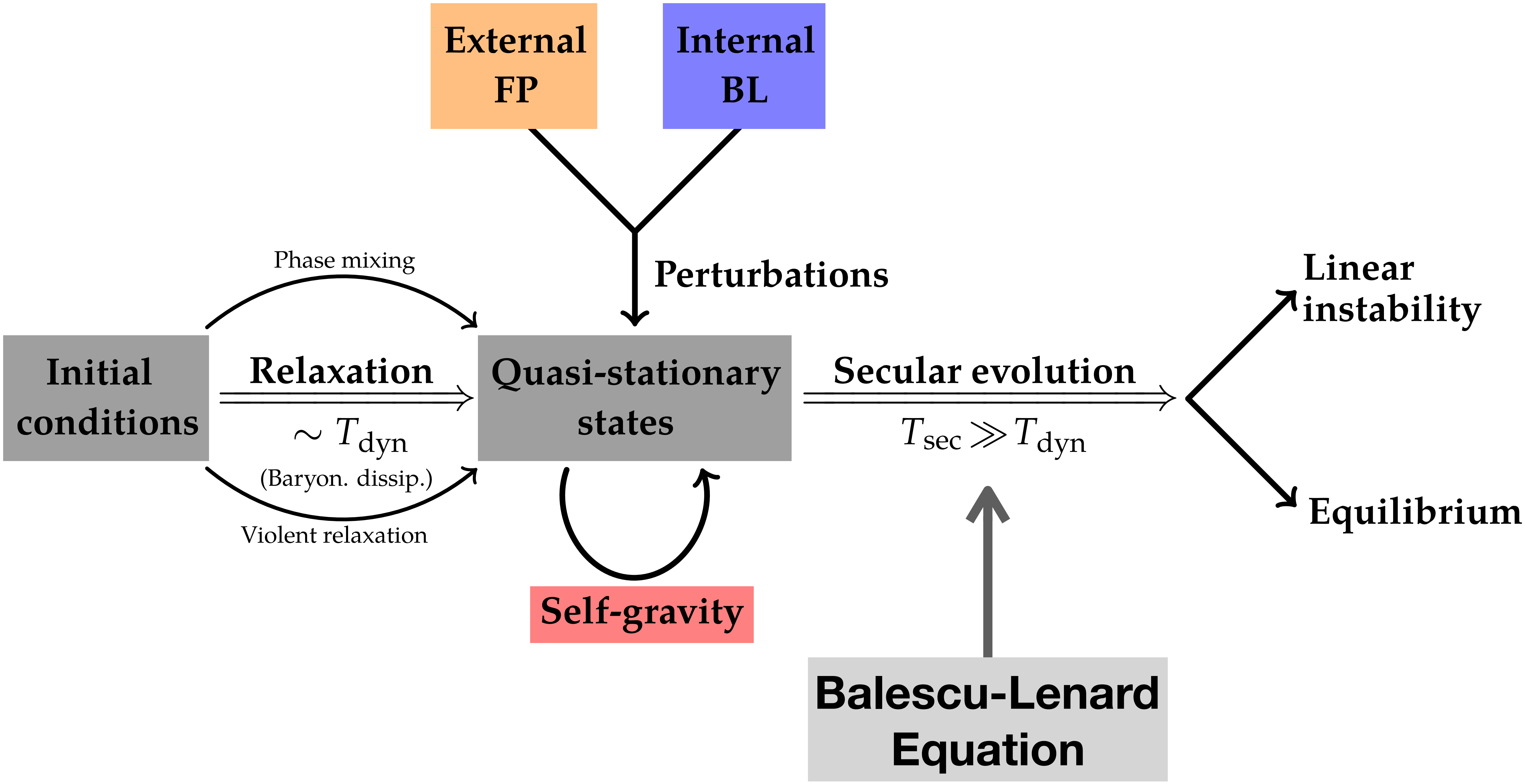
Self-gravitating amplification



Gravitational polarisation essential to

+ *Cause dynamical instabilities*
+ *Induce **dynamical friction** and **mass segregation***
+ ***Accelerate/Slow down** secular evolution*

Typical fate of a self-gravitating system



What is the Balescu-Lenard Eq.?

Balescu-Lenard equation

The master equation for **self-induced orbital relaxation**

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))}{|\,\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))\,|^2} \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} \right) F(\mathbf{J},t)\, F(\mathbf{J}',t) \right]$$

Some properties

$F(\mathbf{J},t)$ Orbital distortion in **action space**

$1/N$ Sourced by **finite-N effects**

$\partial/\partial \mathbf{J}\,\cdot$ Divergence of a **diffusion flux**

$(\mathbf{k},\mathbf{k}')$ Discrete **resonances**

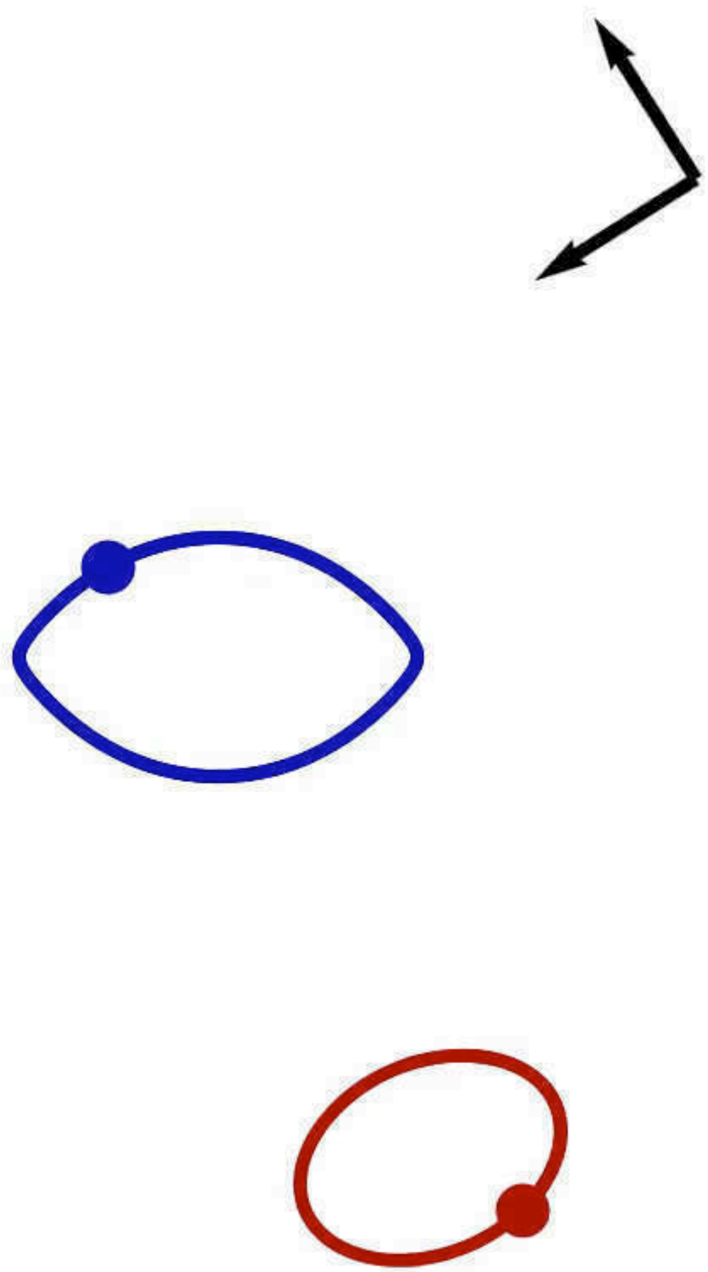
$\int \mathrm{d}\mathbf{J}'$ Scan of **orbital space**

$\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})-\mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))$ **Resonance** cond.

$1/|\,\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)\,|^2$ **Dressed** couplings

Resonant encounters

$$\delta_{\mathrm{D}}(\mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \mathbf{\Omega}(\mathbf{J}'))$$



Collisions are **resonant**, **long-range**, **correlated**

Dressed resonant encounters

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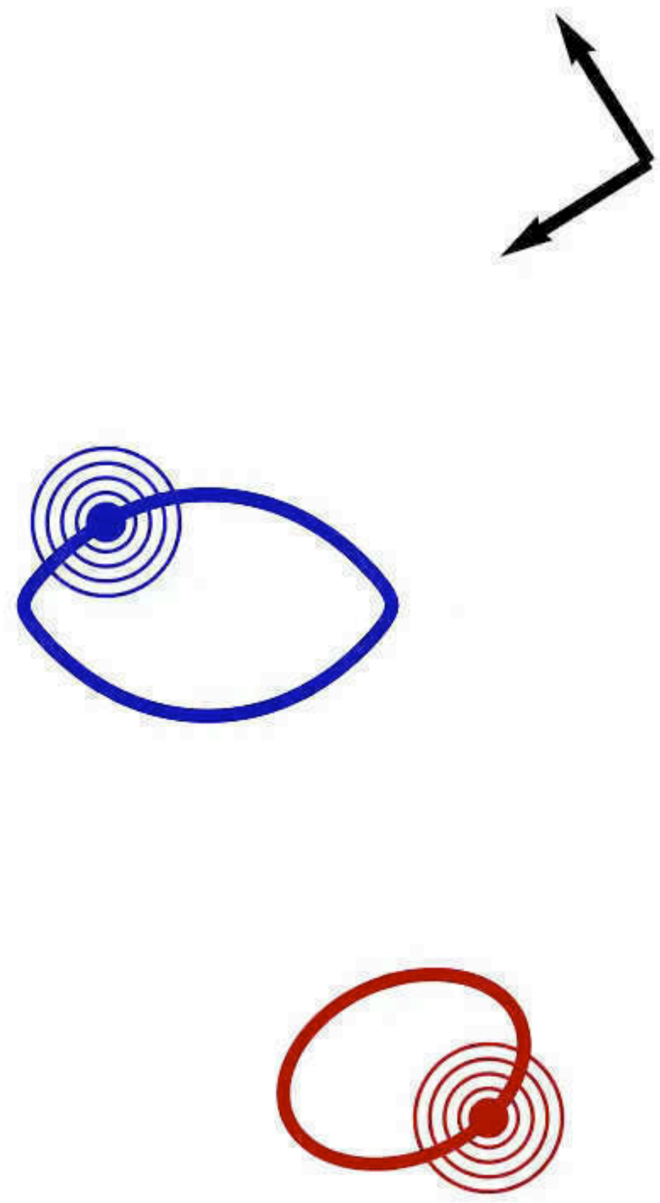


Fluctuations have a **wake**

$$\delta\Phi \rightarrow \frac{\delta\Phi}{|\,\varepsilon(\omega)\,|}$$

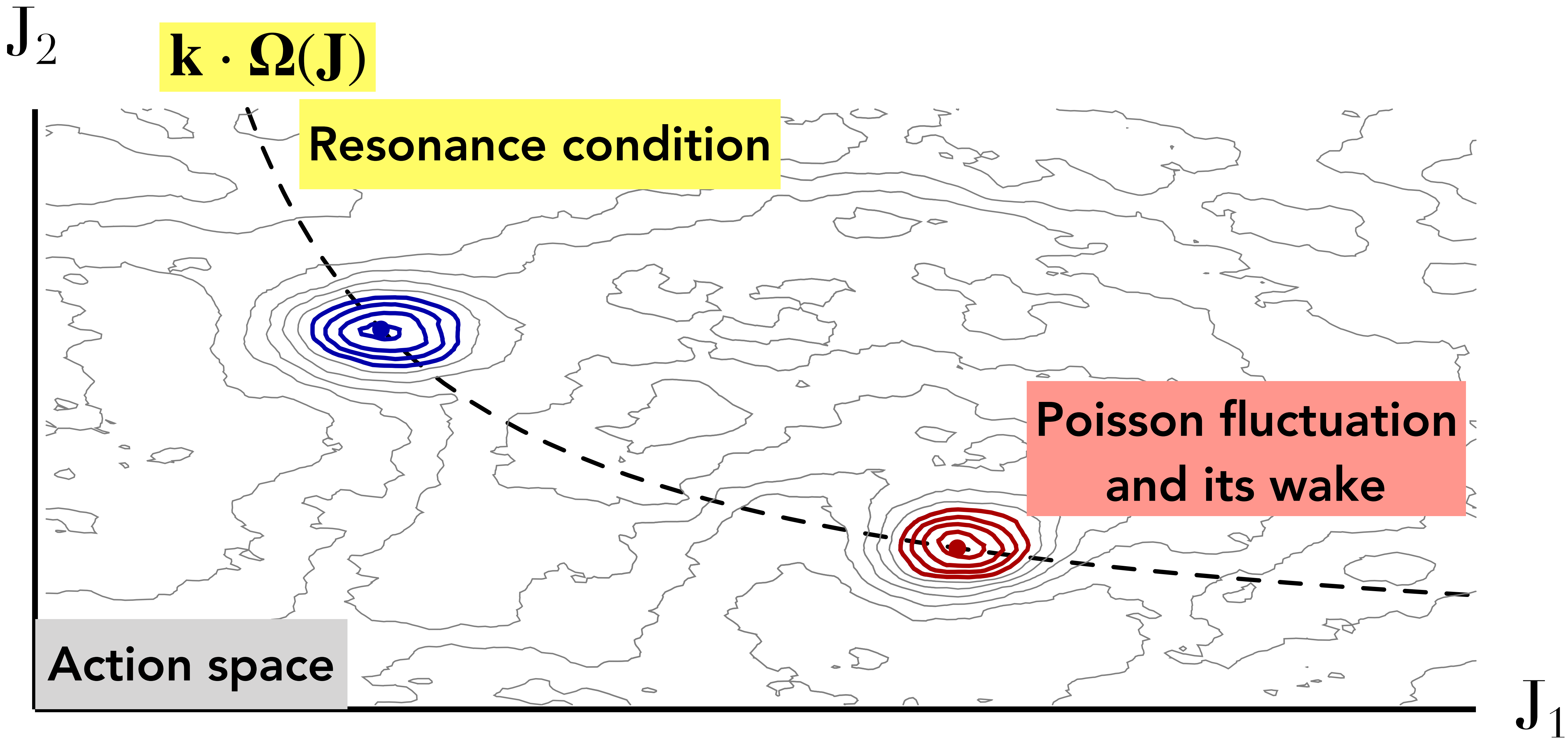
Interactions between **wakes**

$$\mathbf{D}_{\mathrm{diff}}(\mathbf{J}) \rightarrow \frac{\mathbf{D}_{\mathrm{diff}}(\mathbf{J})}{|\,\varepsilon(\omega)\,|^2}$$

Collisions are **resonant**, **long-range**, **correlated,** and **dressed**

Non-local resonances

$$\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))$$



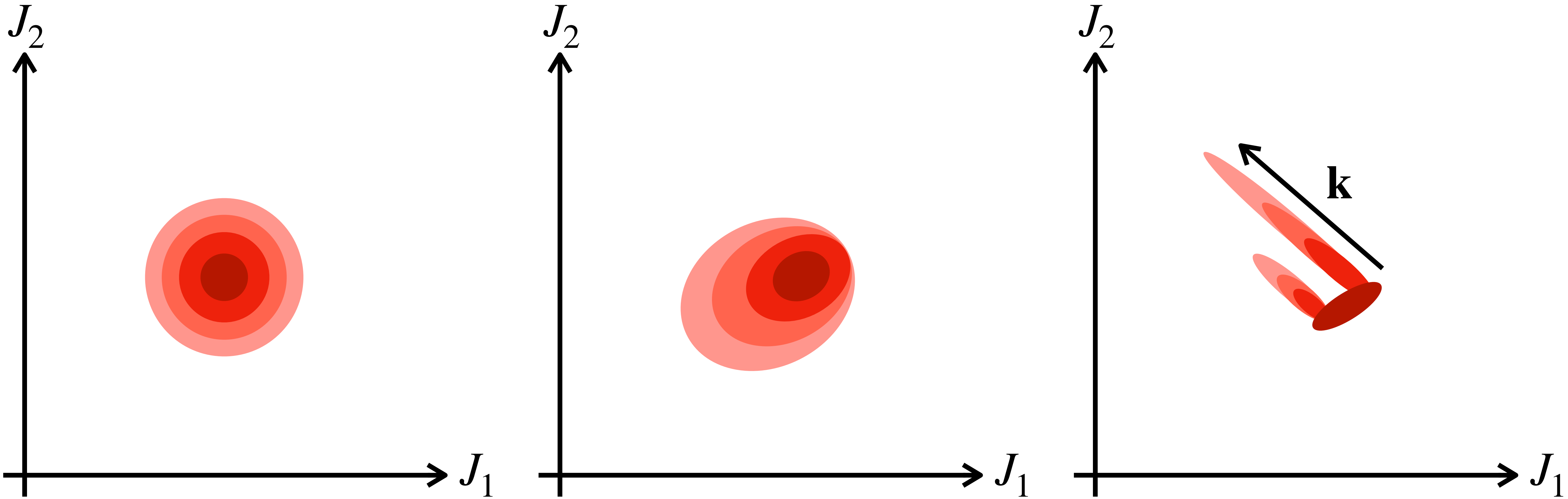
Non-local resonant couplings between **dressed** wakes

Diffusion is anisotropic

Generic **diffusion equation**

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k}} \mathbf{k}\, \mathbf{D}_{\mathbf{k}}(\mathbf{J}) \cdot \frac{\partial F}{\partial \mathbf{J}} \right]$$

Two sources of **anisotropies**



$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[D \, \frac{\partial F}{\partial \mathbf{J}} \right]$$

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[D(\mathbf{J}) \, \frac{\partial F}{\partial \mathbf{J}} \right]$$

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\mathbf{k}\, \mathbf{D}(\mathbf{J}) \cdot \frac{\partial F}{\partial \mathbf{J}} \right]$$

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Fokker-Planck equation

+ **Test particle** of mass m_{t} — $P(\mathbf{J}, t)$
+ **Bath particles** of mass $m_{\mathrm{b}} = M_{\mathrm{tot}}/N$ — $F_{\mathrm{b}}(\mathbf{J}, t)$

$$\frac{\partial P(\mathbf{J},t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))}{|\, \varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}', \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))\,|^2} \times \left(m_{\mathrm{b}}\, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - m_{\mathrm{t}}\, \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} \right) P(\mathbf{J},t)\, F_{\mathrm{b}}(\mathbf{J}',t) \right]$$

Diffusion $m_{\mathrm{b}}\, \mathbf{k}\cdot\dfrac{\partial}{\partial \mathbf{J}}$

Vanishes in the collisionless limit $N \to +\infty$

Sourced **correlations** in the **potential fluctuations**

$$\mathbf{D}_{\mathrm{diff}} \propto \left\langle \delta\Phi(t)\, \delta\Phi(t') \right\rangle$$

Friction $m_{\mathrm{t}}\, \mathbf{k}'\cdot\dfrac{\partial}{\partial \mathbf{J}'}$

$$\mathbf{D}_{\mathrm{fric}} \propto \left\langle \delta P(t)\, \delta\Phi(t') \right\rangle$$

Induces **mass segregation**

Sourced by the **backreaction** of the test particle on the bath

Where does the Balescu-Lenard Eq. come from?

Where does it come from?

Heyvaerts 10

Direct resolution of **BBGKY**

$$\frac{\partial F}{\partial t} = \ldots \quad ; \quad \frac{\partial G_2}{\partial t} = \ldots$$

Chavanis 12

Quasilinear Klimontovich equation

$$\frac{\langle F \rangle}{\partial t} = \ldots \quad ; \quad \frac{\partial \delta F}{\partial t} = \ldots$$

Heyvaerts et al. 17

Fokker-Planck calculation

$$\left\langle \frac{\Delta \mathbf{J}}{\Delta t} \right\rangle \quad ; \quad \left\langle \frac{\Delta \mathbf{J} \otimes \Delta \mathbf{J}}{\Delta t} \right\rangle$$

Functional approach

$$\mathrm{i} \int \mathrm{d}t \, \mathrm{d}\mathbf{w} \, \lambda \left[\frac{\partial F}{\partial t} + \ldots \right]$$

BBGKY and **degenerate systems**

$$\forall \mathbf{J}, \, \mathbf{n} \cdot \mathbf{\Omega}(\mathbf{J}) = 0$$

Stochastic approach and **Novikov theorem**

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \eta(\boldsymbol{\theta}, \mathbf{J}, t)$$

Difficulties

Diffusion in **orbital space** : $F(\mathbf{J}, t)$

Accounting for **collective effects** : $1/ |\, \varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}', \omega) \,|^2$

Timescale decoupling : $\partial \langle F \rangle / \partial t \ll \partial \delta F / \partial t$

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Balescu-Lenard from BBGKY

N **identical particles** of mass $m = \frac{M_{\text{tot}}}{N}$ in phase space $\mathbf{w}_i = (\mathbf{x}_i, \mathbf{v}_i)$

Total specific **Hamiltonian**

$$H_N = \sum_{i=1}^{N} U_{\text{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m\, U(\mathbf{w}_i, \mathbf{w}_j)$$

3D self-gravitating systems

$$U_{\text{ext}} = \frac{|\mathbf{v}|^2}{2}$$
$$U = -\frac{G}{|\mathbf{x} - \mathbf{x}'|}$$

System characterised by the **N-body PDF** $P_N(\mathbf{w}_1, \ldots, \mathbf{w}_N, t)$

Continuity equation in phase space

$$\frac{\partial P_N}{\partial t} + \sum_i \frac{\partial}{\partial \mathbf{w}_i} \cdot \left(P_N\, \dot{\mathbf{w}}_i \right) = 0$$

Exact Liouville equation

$$\frac{\partial P_N}{\partial t} + \left[P_N, H_N \right]_N = 0$$

BBGKY hierarchy

Reduced DFs

$$F_n(\mathbf{w}_1, \ldots, \mathbf{w}_n, t) = m^n \frac{N!}{(N-n)!} \int \mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N \, P_N(\mathbf{w}_1, \ldots, \mathbf{w}_N, t)$$

BBGKY hierarchy

$$\frac{\partial F_n}{\partial t} + \left[F_n, H_n\right]_n + \int \mathrm{d}\mathbf{w}_{n+1} \left[F_{n+1}, \delta H_{n+1}\right]_n = 0$$

With

$$H_n = \sum_{i=1}^{n} U_{\mathrm{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m \, U(\mathbf{w}_i, \mathbf{w}_j)$$

n-body system

$$\delta H_{n+1} = \sum_{i=1}^{n} U(\mathbf{w}_i, \mathbf{w}_{n+1})$$

Interactions with (n+1)

BBGKY at 1/N

Cluster representation of the DFs

$$\begin{cases} F_2(\mathbf{w},\mathbf{w}') = F_1(\mathbf{w})\,F_1(\mathbf{w}') & + \; G_2(\mathbf{w},\mathbf{w}') \\ F_3(\mathbf{w},\mathbf{w}',\mathbf{w}'') = \ldots & + \; G_3(\mathbf{w},\mathbf{w}',\mathbf{w}'') \end{cases} \Longrightarrow \begin{cases} G_2 \sim 1/N \\ G_3 \sim 1/N^2 \end{cases}$$

Truncation at **order 1/N**: 2 dynamical quantities

$F(\mathbf{w},t)$ **1-body DF**

$G(\mathbf{w},\mathbf{w}',t)$ **2-body correlation**

BBGKY - 1

$$\frac{\partial F}{\partial t} + \big[F, H_0\big]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}'\, \big[G, U(\mathbf{w},\mathbf{w}')\big]_{\mathbf{w}} = 0$$

BBGKY - 2

$$\frac{\partial G}{\partial t} + \big[G, H_0\big]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}''\, G(\mathbf{w}',\mathbf{w}'')\, \big[F(\mathbf{w}), U(\mathbf{w},\mathbf{w}'')\big]_{\mathbf{w}}$$
$$+ m\, \big[F(\mathbf{w})\,F(\mathbf{w}'), U(\mathbf{w},\mathbf{w}')\big]_{\mathbf{w}} + (\mathbf{w} \leftrightarrow \mathbf{w}') = 0$$

BBGKY - 1

$$\frac{\partial F}{\partial t} + \left[F, H_0\right]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}' \left[G, U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}} = 0$$

$\left[F, H_0\right]_{\mathbf{w}}$ Mean-field advection

$\int \mathrm{d}\mathbf{w}' \left[G, U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}}$ **Collision** term

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BBGKY - 2

$$\frac{\partial G}{\partial t} + \left[G, H_0\right]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}'' \, G(\mathbf{w}', \mathbf{w}'') \left[F(\mathbf{w}), U(\mathbf{w}, \mathbf{w}'')\right]_{\mathbf{w}}$$
$$+ m \left[F(\mathbf{w}) \, F(\mathbf{w}'), U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}} + (\mathbf{w} \leftrightarrow \mathbf{w}') = 0$$

$\left[G, H_0\right]_{\mathbf{w}}$ Mean-field advection

$\int \mathrm{d}\mathbf{w}'' \, G(\mathbf{w}', \mathbf{w}'') \left[F(\mathbf{w}), U(\mathbf{w}, \mathbf{w}'')\right]_{\mathbf{w}}$ Collective effects

$\left[F(\mathbf{w}) \, F(\mathbf{w}'), U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}}$ 1-body DF sourcing

How to solve BBGKY

Adiabatic approximation

i.e. evolution along **quasi-stationary states**

$$F = F(\mathbf{J}, t) \;\; ; \;\; H_0 = H_0(\mathbf{J}, t) \quad \Longrightarrow \quad \big[F_0(\mathbf{J}) \, , H_0(\mathbf{J})\big]_{\mathbf{w}} = 0$$

Mean-field equilibrium

Timescale separation

$$\frac{\partial G}{\partial t} + \big[G, H_0\big]_{\mathbf{w}} + \big(\ldots\big) = 0$$

$$\frac{\partial F}{\partial t} = - \int \mathrm{d}\mathbf{w}' \, \big[G, U(\mathbf{w}, \mathbf{w}')\big]_{\mathbf{w}}$$

Collision operator

$$\longrightarrow \begin{cases} T_G \simeq T_{\mathrm{dyn}} \\ T_F \simeq N \times T_G \end{cases}$$

Bogoliubov's Ansatz

$$\frac{\partial G}{\partial t} = \mathrm{BBGKY}_2\big[F = \mathrm{cst} \, , G\big]$$

$$\frac{\partial F}{\partial t} = \mathrm{BBGKY}_1\big[F \, , G(t \to +\infty)\big]$$

The dynamics of correlations

Time evolution of the **correlations**

$$\frac{\partial G(\mathbf{w},\mathbf{w}')}{\partial t} + V_{\mathbf{w}}(G) + V_{\mathbf{w}'}(G) = S(\mathbf{w},\mathbf{w}')$$

Vlasov operator   Source term

Linearised Vlasov operator

$$V_{\mathbf{w}}(f(\mathbf{w})) = \big[f(\mathbf{w})\,, H_0(\mathbf{w})\big]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}' \big[f(\mathbf{w}')F_0(\mathbf{w}), U(\mathbf{w},\mathbf{w}')\big]_{\mathbf{w}}$$

Mean field   Collective effects

Solved using **Green's functions**

$$G(\mathbf{w},\mathbf{w}',t) = \int \mathrm{d}\tilde{\mathbf{w}}\,\mathrm{d}\tilde{\mathbf{w}}'\,\mathrm{Green}\big[\mathbf{w},\mathbf{w}'\,|\,\tilde{\mathbf{w}},\tilde{\mathbf{w}}',t\big]\,S(\tilde{\mathbf{w}},\tilde{\mathbf{w}}',0)$$

Green's function   Time-independent

Miracle: Vlasov operator acts **independently** on $(\mathbf{w},\mathbf{w}')$

$$\mathrm{Green}\big[\mathbf{w},\mathbf{w}'\,|\,\tilde{\mathbf{w}},\tilde{\mathbf{w}}',t\big] = \mathrm{Green}\big[\mathbf{w}\,|\,\tilde{\mathbf{w}},t\big]\,\mathrm{Green}\big[\mathbf{w}'\,|\,\tilde{\mathbf{w}}',t\big]$$

Separability

Where does it come from?

Heyvaerts 10

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Functional approach

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Diffusion in **orbital space** : $F(\mathbf{J}, t)$

Accounting for **collective effects** : $1/|\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}', \omega)|^2$

Timescale decoupling : $\partial \langle F \rangle / \partial t \ll \partial \delta F / \partial t$

Balescu-Lenard via Klimontovich

Describing one **realisation** in **phase space** $\mathbf{w} = (\mathbf{x}, \mathbf{v})$

Discrete DF

$$F_{\mathrm{d}}(\mathbf{w}, t) = \sum_{i=1}^{N} m\, \delta_{\mathrm{D}}(\mathbf{w} - \mathbf{w}_i(t))$$

3D gravitational systems

$$U_{\mathrm{ext}} = \frac{|\mathbf{v}|^2}{2}$$

$$U = -\frac{G}{|\mathbf{x} - \mathbf{x}'|}$$

Discrete Hamiltonian

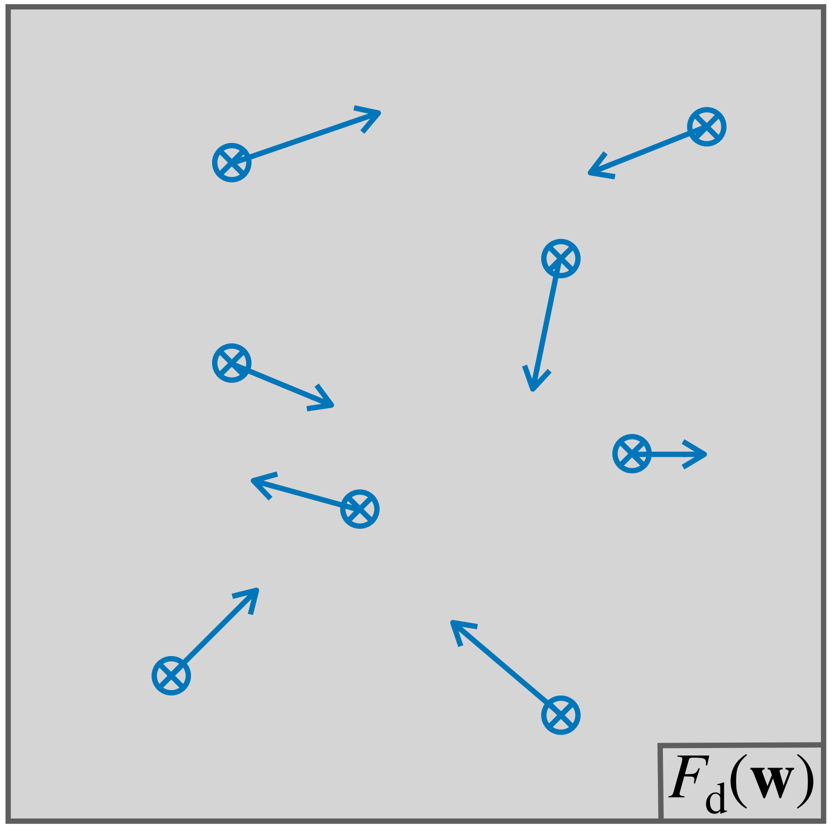
$$H_{\mathrm{d}}(\mathbf{w}, t) = U_{\mathrm{ext}}(\mathbf{w}) + \int \mathrm{d}\mathbf{w}'\, F_{\mathrm{d}}(\mathbf{w}', t)\, U(\mathbf{w}, \mathbf{w}')$$

Continuity equation in phase space

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \cdot \left(F_{\mathrm{d}}\, \dot{\mathbf{w}} \right) = 0$$

Exact **Klimontovich** equation

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + \left[F_{\mathrm{d}}, H_{\mathrm{d}} \right] = 0$$

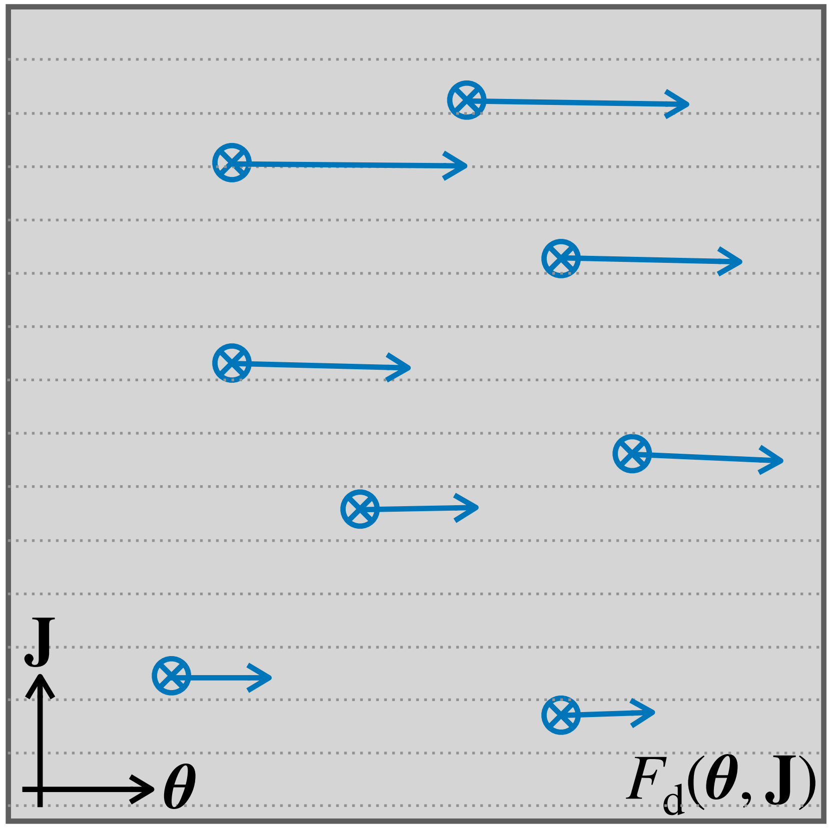


Phase space

Solving Klimontovich

Perturbative expansion

$$\begin{cases} F_{\mathrm{d}} = F_0 + \delta F \;\; \text{with} \;\; \langle \delta F \rangle = 0, \\ H_{\mathrm{d}} = H_0 + \delta H \;\; \text{with} \;\; \langle \delta H \rangle = 0. \end{cases}$$



Angle-Action space

Adiabatic approximation

$$\begin{cases} F_0 = F_0(\mathbf{J}, t), \\ H_0 = H_0(\mathbf{J}, t) \,. \end{cases}$$

Quasi-linear evolution equations

$$\frac{\partial F_0}{\partial t} = -\left\langle \left[\delta F, \delta H\right] \right\rangle$$

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] + \left[F_0, \delta H\right] = 0$$

Timescale separation

$$\begin{cases} T_{\delta F} \simeq T_{\mathrm{dyn}} \\ T_{F_0} \simeq \left(\sqrt{N}\right)^2 \times T_{\delta F} \end{cases}$$

Dynamics of fluctuations

Fast evolution of **perturbations** (Linearised Klimontovich Eq.)

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] + \left[F_0, \delta H\right] = 0$$

$\left[\delta F, H_0\right]$ Mean-field advection

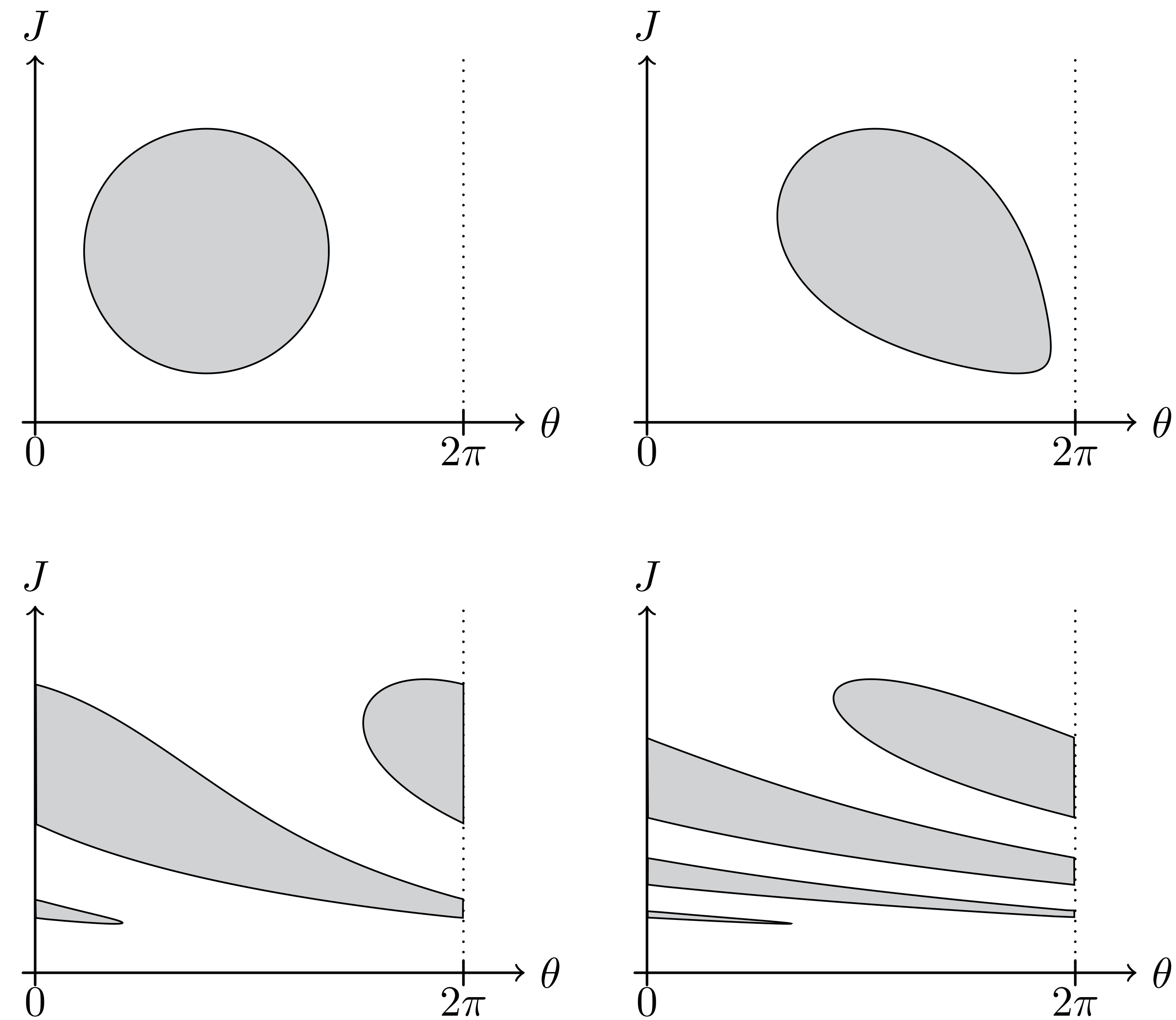
$\left[F_0, \delta H\right]$ Collective effects

Self-consistent amplification

$$\delta H = \delta H\left[\delta F\right]$$

Timescale separation

$$\begin{cases} F_0(\mathbf{J}) = \text{cst} \\ H_0(\mathbf{J}) = \text{cst} \end{cases}$$



Phase Mixing

Solving for the fluctuations

Linear amplification

$$\delta \hat{F}_{\mathbf{k}}(\mathbf{J},\omega) = -\frac{\delta F_{\mathbf{k}}(\mathbf{J},0)}{\mathrm{i}(\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))} - \frac{\mathbf{k}\cdot\partial F_0/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})}\,\delta \hat{H}_{\mathbf{k}}(\mathbf{J},\omega)$$

Bare noise

Self-consistent amplification

with the **self-consistency**

$$\delta H(\mathbf{w},t) = \int \mathrm{d}\mathbf{w}'\,\delta F(\mathbf{w}',t)\,U(\mathbf{w},\mathbf{w}')$$

Generic form of a **Fredholm equation**

$$\big[\delta H(\mathbf{J})\big]_{\mathrm{dressed}} = \big[\delta H(\mathbf{J})\big]_{\mathrm{bare}} + \int \mathrm{d}\mathbf{J}'\,M(\mathbf{J},\mathbf{J}')\,\big[\delta H(\mathbf{J}')\big]_{\mathrm{dressed}}$$

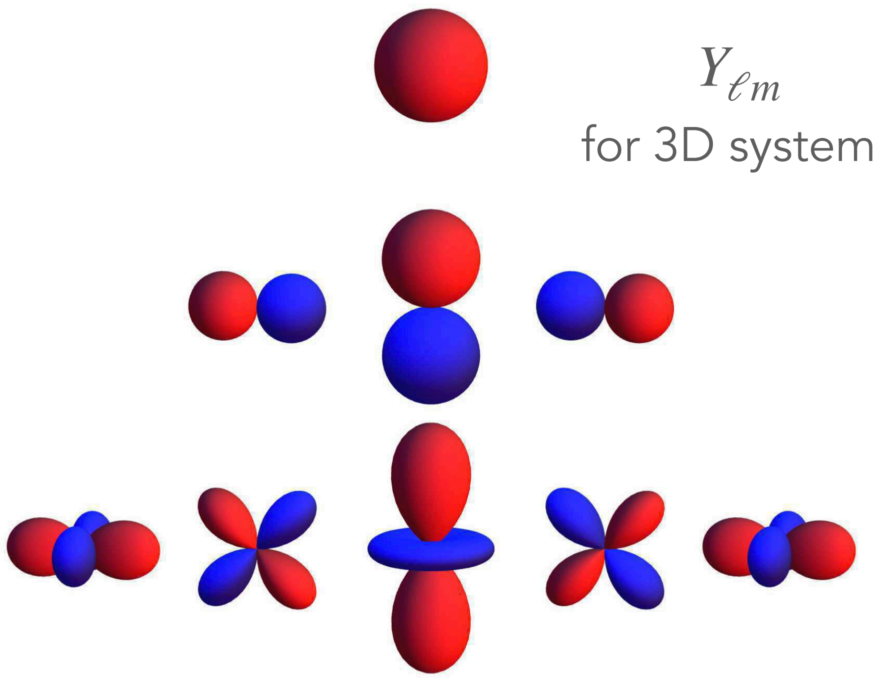
Amplification kernel

Dressing of perturbations

$$\big[\delta H(\omega)\big]_{\mathrm{dressed}} \simeq \frac{[\delta H(\omega)]_{\mathrm{bare}}}{1 - M(\omega)} = \frac{[\delta H(\omega)]_{\mathrm{bare}}}{|\,\varepsilon(\omega)\,|}$$

Basis method $\big(\psi^{(p)}(\mathbf{w}), \rho^{(p)}(\mathbf{w})\big)$

$$\begin{cases} \psi^{(p)}(\mathbf{w}) = \displaystyle\int \mathrm{d}\mathbf{w}'\, U(\mathbf{w},\mathbf{w}')\, \rho^{(p)}(\mathbf{w}'), \\ \displaystyle\int \mathrm{d}\mathbf{w}\, \psi^{(p)}(\mathbf{w})\, \rho^{(q)*}(\mathbf{w}) = -\,\delta_{pq}\,. \end{cases}$$



“**Separable**” pairwise interaction

$$U(\mathbf{w},\mathbf{w}') = -\sum_p \psi^{(p)}(\mathbf{w})\, \psi^{(p)*}(\mathbf{w}')$$

Plasmas

$$U(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

$$\simeq \int \frac{\mathrm{d}\mathbf{k}}{|\mathbf{k}|^2}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}$$

Galaxies

$$\Delta\Phi = 4\pi G\rho$$

Poisson equation

Linear response theory

$$\left[\delta H(\omega)\right]_{\text{dressed}} = \frac{[\delta H(\omega)]_{\text{bare}}}{|\,\varepsilon(\omega)\,|}$$

$$\varepsilon_{pq}(\omega) = 1 - \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{k}\cdot\partial F_0/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})} \, \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \, \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Dielectric function

$$\varepsilon_{pq}(\omega)$$

Two limits

$\varepsilon_{pq}(\omega) \simeq 0$ ***Cold*** *regime*

$\varepsilon_{pq}(\omega) \simeq 1$ ***Hot*** *regime*

Some properties

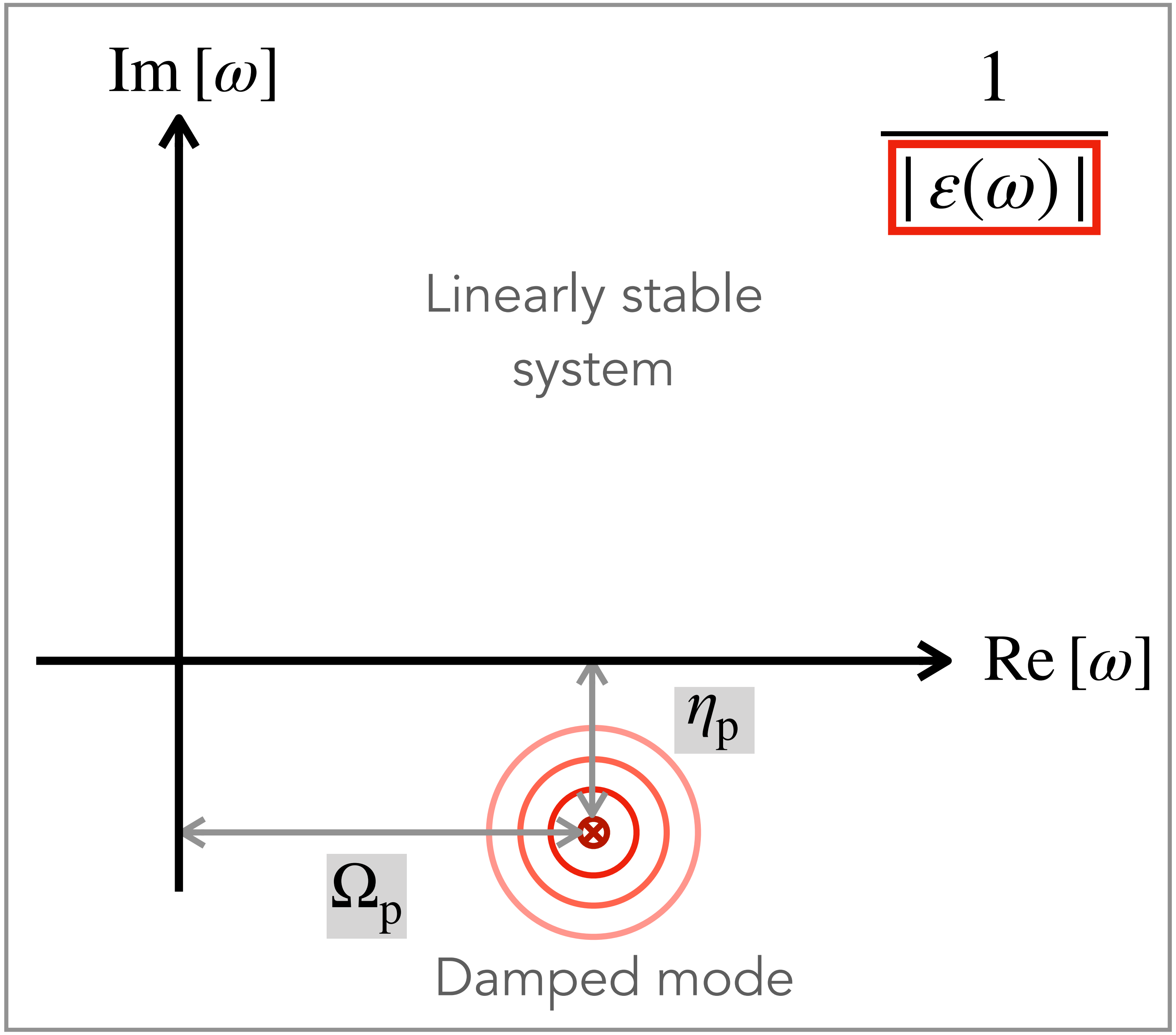
$\sum_{\mathbf{k}}$ *Sum over resonances*

$\int \mathrm{d}\mathbf{J}$ *Scan over orbital space*

$\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})$ *Resonant int.*

$\psi^{(p)} = \int \mathrm{d}\mathbf{w}' \, U \, \rho^{(p)}$ *Long-range int.*

Dielectric function



Susceptibility

$$\frac{1}{|\varepsilon(\Omega_{\mathrm{p}})|} \gg 1$$

Thermalisation

$$\left[\delta H(t)\right]_{\mathrm{trans.}} \simeq \mathrm{e}^{-\eta_{\mathrm{p}} t}$$

Dressed long-term diffusion

Secular evolution equation

$$\frac{\partial F_0}{\partial t} = - \left\langle \left[\delta F, \delta H \right] \right\rangle$$

Dressing comes twice

$$\left[\delta H\right]_{\text{dressed}} = \frac{\left[\delta H\right]_{\text{bare}}}{\boxed{|\,\varepsilon(\omega)\,|}} \longrightarrow \frac{\partial F_0}{\partial t} \simeq \frac{|\,\delta H\,|^2_{\text{bare}}}{\boxed{|\,\varepsilon(\omega)\,|^2}}$$

Bare Poisson shot noise

Relaxation time

$$|\,\delta H\,|_{\text{bare}} \simeq \frac{1}{\sqrt{N}} \longrightarrow T_{\text{relax}} \simeq \boxed{|\,\varepsilon\,|^2} N \, T_{\text{dyn}}$$

Collective effects can **drastically accelerate** orbital heating, in particular on **large scales**

Balescu-Lenard equation

The master equation for **self-induced orbital relaxation**

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))}{|\,\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))\,|^2} \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right) F(\mathbf{J},t)\, F(\mathbf{J}',t) \right]$$

Some properties

- $F(\mathbf{J},t)$ Orbital distortion in **action space**
- $1/N$ Sourced by **finite-N effects**
- $\partial/\partial\mathbf{J}\,\cdot$ Divergence of a **diffusion flux**
- $(\mathbf{k},\mathbf{k}')$ Discrete **resonances**
- $\int \mathrm{d}\mathbf{J}'$ Scan of **orbital space**
- $\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))$ **Resonance** cond.
- $1/|\,\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)\,|^2$ **Dressed** couplings

Plasmas	Galaxies

Orbital coordinates

Plasmas	Galaxies
$(\mathbf{x}, \mathbf{v})$	$(\boldsymbol{\theta}, \mathbf{J})$

Basis decomposition

Plasmas	Galaxies
$U(\mathbf{x}, \mathbf{x}') \propto \int \frac{\mathrm{d}\mathbf{k}}{\mathbf{k}^2}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$	$U(\mathbf{w}, \mathbf{w}') = -\sum_p \psi^{(p)}(\mathbf{w})\, \psi^{(p)*}(\mathbf{w}')$

Dielectric function

Plasmas	Galaxies
$1 - \frac{1}{\mathbf{k}^2} \int \mathrm{d}\mathbf{v} \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}$	$\delta_{pq} - \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J})\psi_{\mathbf{k}}^{(q)}(\mathbf{J})$

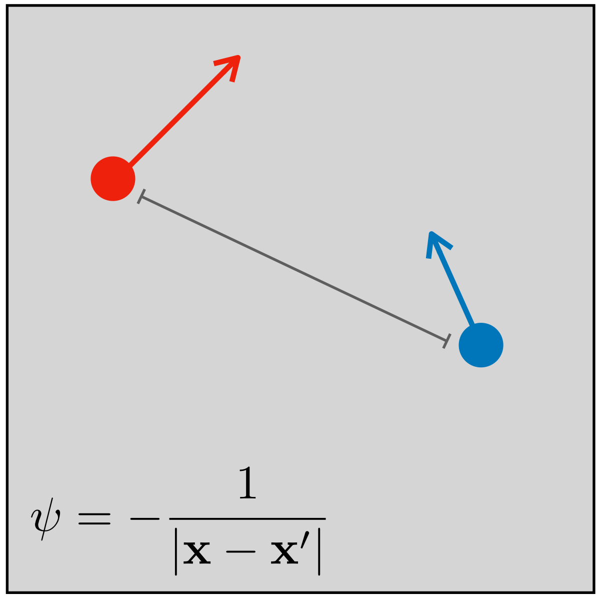
Resonance condition

Plasmas	Galaxies
$\delta_{\mathrm{D}}\big(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')\big)$	$\delta_{\mathrm{D}}\big(\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}')\big)$

Does
the Balescu-Lenard Eq.
work?

Long-range interacting systems are ubiquitous

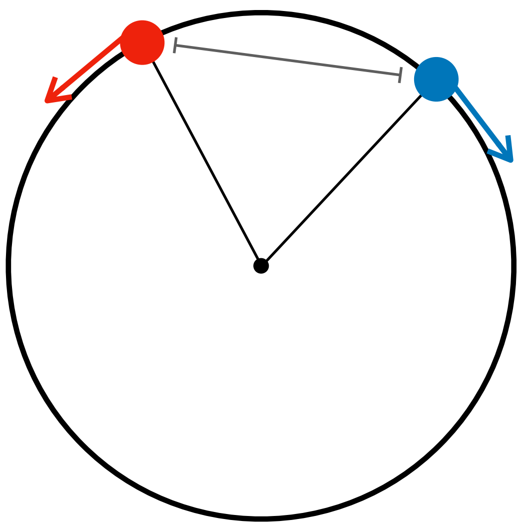
Homogeneous systems



d=3, homogeneous

Hamiltonian Mean Field Model

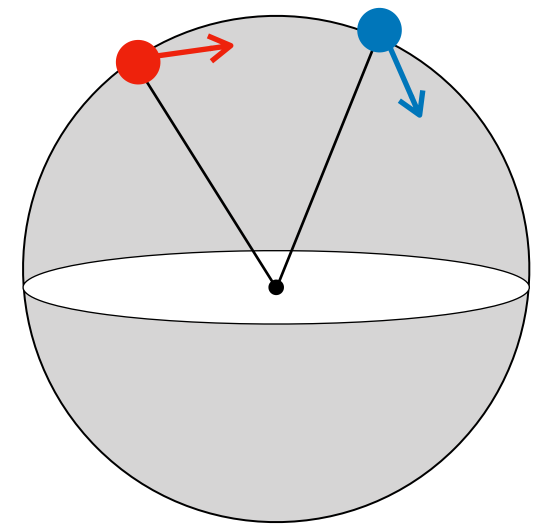
$\psi = -\cos(\theta - \theta')$



d=1, inhomogeneous

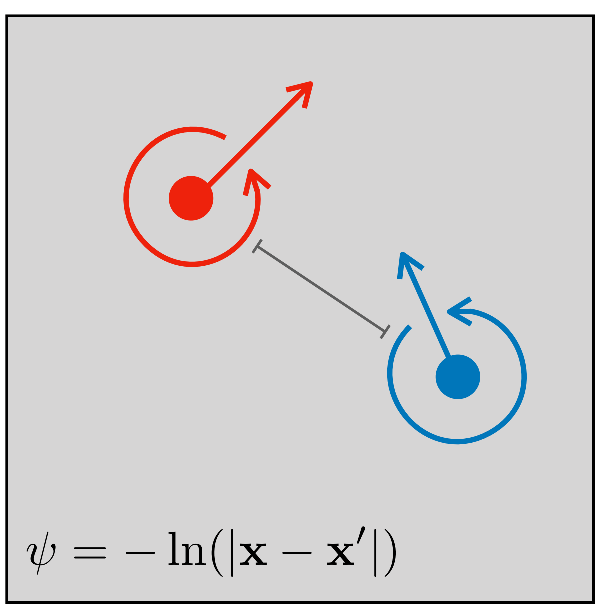
Vector Resonant Relaxation

$\psi = -V(\mathbf{s}\cdot\mathbf{s}')$



d=1, inhomogeneous, degenerate

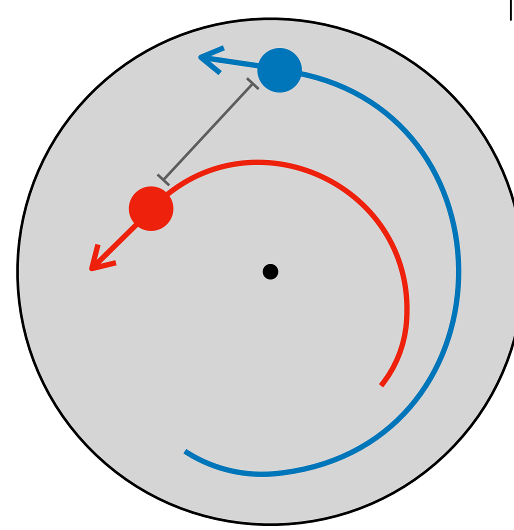
2D hydrodynamics



d=2, inhomogeneous

Self-gravitating discs

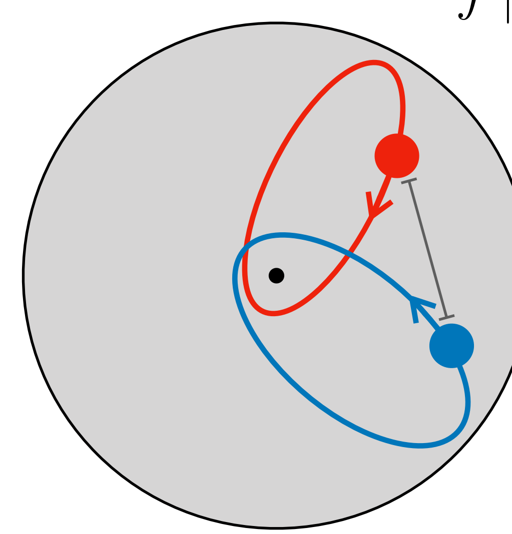
$\psi = -\dfrac{1}{|\mathbf{x} - \mathbf{x}'|}$



d=2, inhomogeneous

Scalar Resonant Relaxation

$\psi = -\displaystyle\int \frac{\mathrm{d}\theta \mathrm{d}\theta'}{|\mathbf{x} - \mathbf{x}'|}$



d=2, inhomogeneous, degenerate

The diversity of long-range interacting systems

Small dimension		Large dimension	
$d = 1$	Galactic Nuclei	$d = 2$	Globular clusters
Homogeneous		**Inhomogeneous**	
$(\mathbf{x}, \mathbf{v})$	Plasmas	$(\boldsymbol{\theta}, \mathbf{J})$	Galaxies
Hot		**Cold**	
$\dfrac{1}{\mid \varepsilon(\omega) \mid} \simeq 1$	Dark matter halo	$\dfrac{1}{\mid \varepsilon(\omega) \mid} \gg 1$	Galactic discs
Non-degenerate		**Degenerate**	
No global resonance	Discs	$\forall \mathbf{J},\, \mathbf{n} \cdot \mathbf{\Omega}(\mathbf{J}) = 0$	Keplerian systems

Balescu-Lenard: A numerical nightmare

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J},t)$$

Balescu-Lenard equation

$$\mathbf{F}(\mathbf{J},t) = \sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))}{|\,\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))\,|^2} \times \left(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\right) F(\mathbf{J})\, F(\mathbf{J}')$$

Diffusion flux

$$\frac{1}{\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)} = \sum_{p,q} \psi^{(p)}_{\mathbf{k}}(\mathbf{J})\, \mathbf{E}^{-1}_{pq}(\omega)\, \psi^{(q)*}_{\mathbf{k}'}(\mathbf{J}')$$

Dressed susceptibility coefficients

$$\mathbf{E}_{pq}(\omega) = \delta_{pq} - \mathbf{M}_{pq}(\omega)$$

Dielectric matrix

$$\mathbf{M}_{pq}(\omega) = \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})} \, \psi^{(p)*}_{\mathbf{k}}(\mathbf{J})\, \psi^{(q)}_{\mathbf{k}}(\mathbf{J})$$

Response matrix

$$\psi^{(p)}_{\mathbf{k}}(\mathbf{J}) = \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d} \, \psi^{(p)}\big(\mathbf{x}[\boldsymbol{\theta},\mathbf{J}]\big)\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}$$

Basis elements

A numerical nightmare

With also:

- + Integral over $\mathrm{d}\boldsymbol{\theta}$
- + (Double) integral over $\mathrm{d}\mathbf{J}$
- + (Triple) sum over $\mathbf{k}$
- + (Double) sum over (p, q)
- + Matrix inversion
- + Resonant denominator
- + Resonance condition

$$\mathbf{F}(\mathbf{J}, t) = \sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))}{|\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}', \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))|^2} \times \left(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\right) F(\mathbf{J})\, F(\mathbf{J}')$$

Diffusion flux

$$\frac{1}{\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}', \omega)} = \sum_{p,q} \psi^{(p)}_{\mathbf{k}}(\mathbf{J})\, \mathbf{E}^{-1}_{pq}(\omega)\, \psi^{(q)*}_{\mathbf{k}'}(\mathbf{J}')$$

Dressed susceptibility coefficients

$$\mathbf{E}_{pq}(\omega) = \delta_{pq} - \mathbf{M}_{pq}(\omega)$$

Dielectric matrix

$$\mathbf{M}_{pq}(\omega) = \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})} \psi^{(p)*}_{\mathbf{k}}(\mathbf{J})\, \psi^{(q)}_{\mathbf{k}}(\mathbf{J})$$

Response matrix

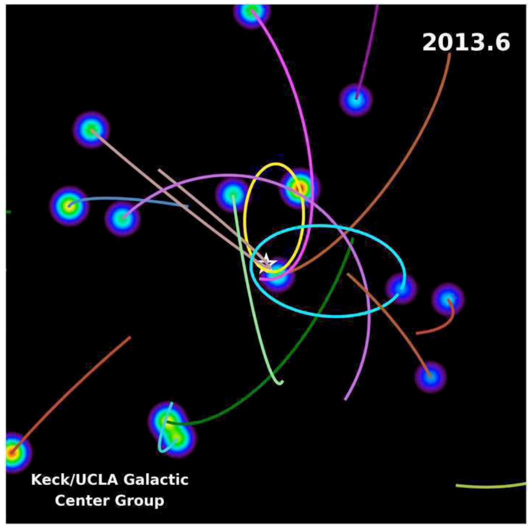
$$\psi^{(p)}_{\mathbf{k}}(\mathbf{J}) = \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d} \psi^{(p)}\big(\mathbf{x}[\boldsymbol{\theta}, \mathbf{J}]\big)\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}$$

Basis elements

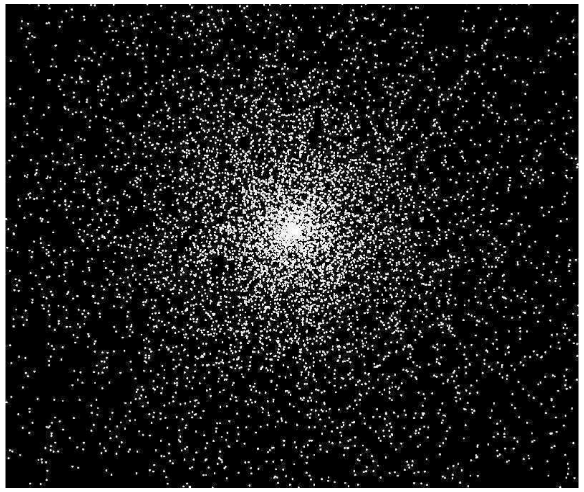
Does it work?

Galactic discs

Galactic nuclei



Globular clusters



Does it work?

Galactic discs

$$\frac{1}{|\,\varepsilon(\omega)\,|} \gg 1$$

Dynamically cold system

Globular clusters

$$(\mathbf{k},\mathbf{k}') \in [\![1,+\infty]\!]$$

Large number of resonances

Galactic nuclei

$$U(\mathbf{w},\mathbf{w}') \mapsto \overline{U} = \int \frac{\mathrm{d}\theta}{2\pi}\frac{\mathrm{d}\theta'}{2\pi}\, U$$

Orbit-averaged interactions

Does it work?

Galactic discs

$$\frac{1}{|\varepsilon(\omega)|} \gg 1$$

Dynamically cold system

Galactic nuclei

$$U(\mathbf{w}, \mathbf{w}') \mapsto \overline{U} = \int \frac{\mathrm{d}\theta}{2\pi} \frac{\mathrm{d}\theta'}{2\pi} U$$

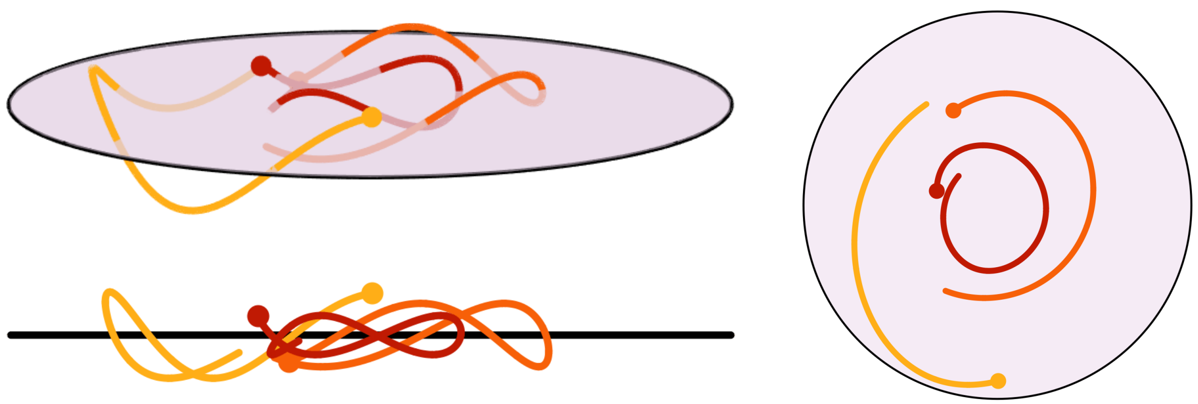
Orbit-averaged interactions

Globular clusters

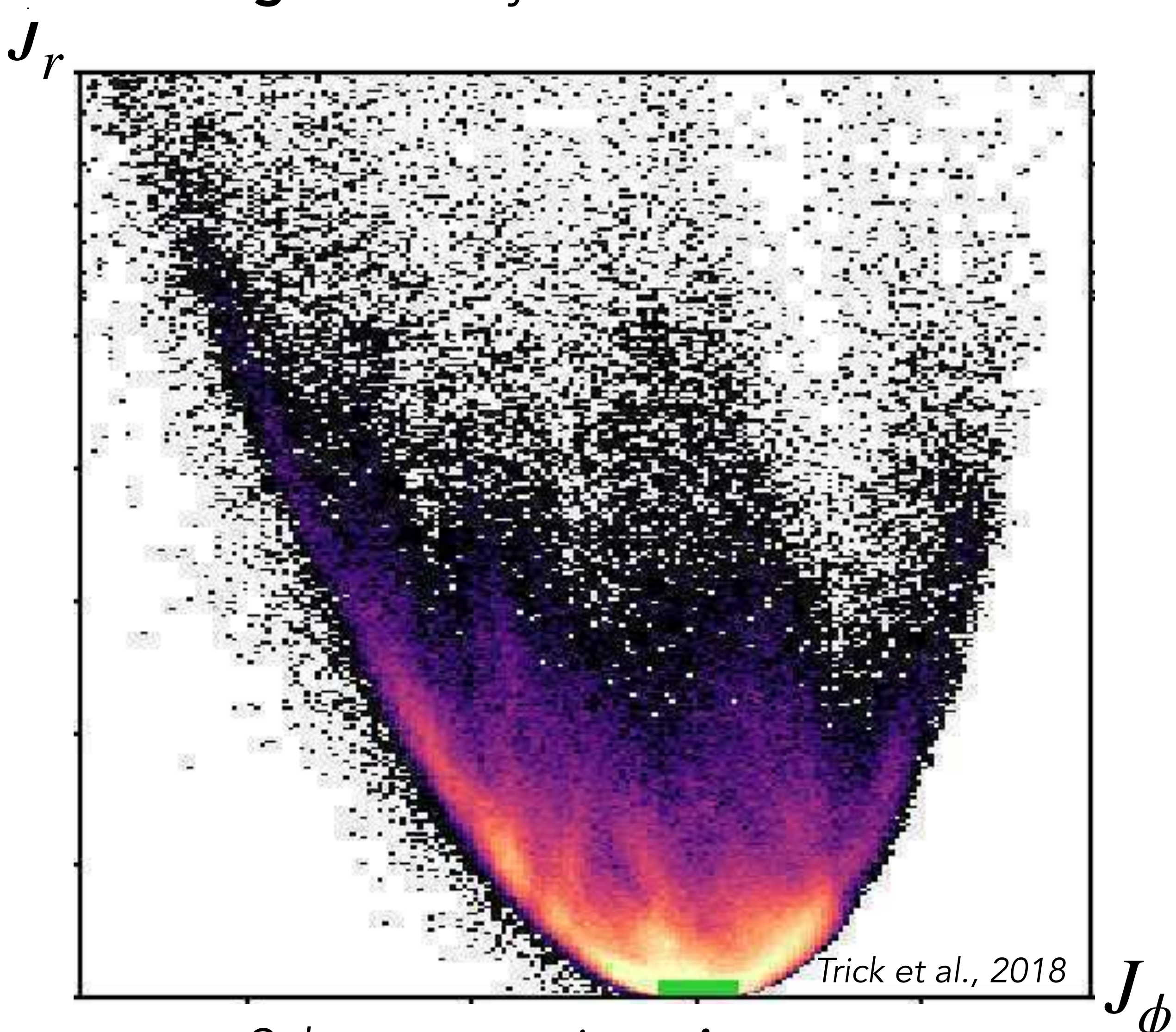
$$(\mathbf{k}, \mathbf{k}') \in [\![1, +\infty]\!]$$

Large number of resonances

Galactic discs



Inhomogeneous system and intricate orbits

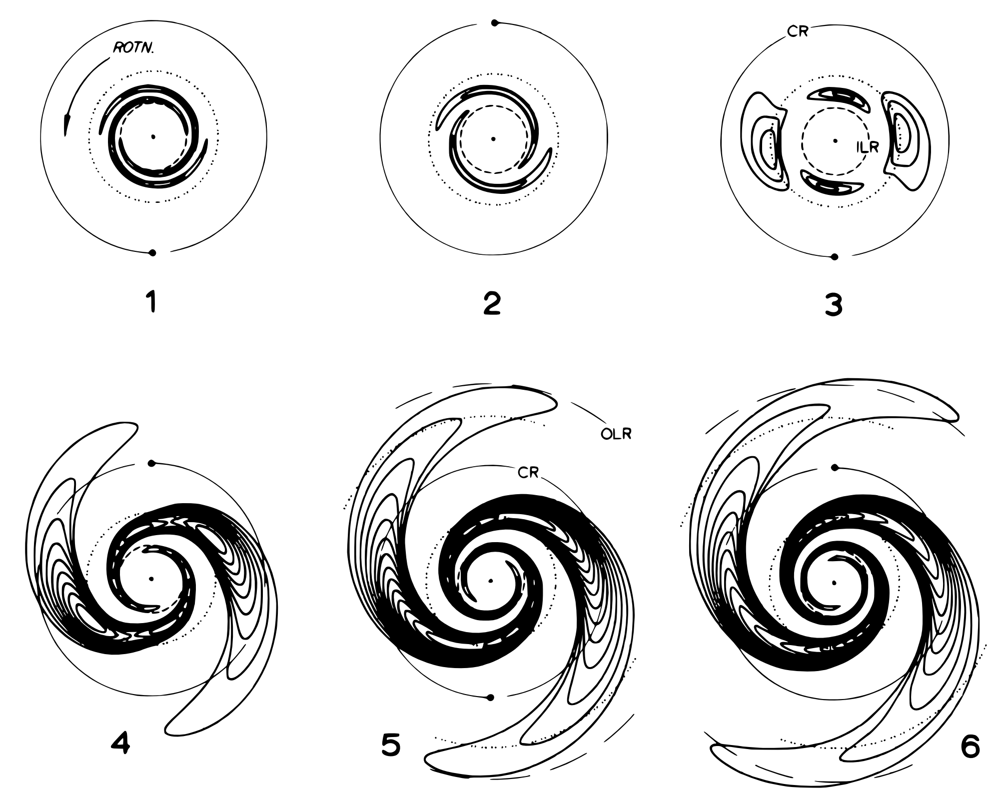


*Sub-structures in **action space**,*
as observed by GAIA

How do stars diffuse in **galactic discs**?

+ **Galactic archeology**
+ Formation of **spiral arms/bars**
+ Local **velocity anisotropies**
+ Disc **thickening**
+ Stellar **streams**

Swing amplification in **cold** discs



Toomre, 1981

$$\frac{1}{|\varepsilon(\omega)|} \simeq 30$$

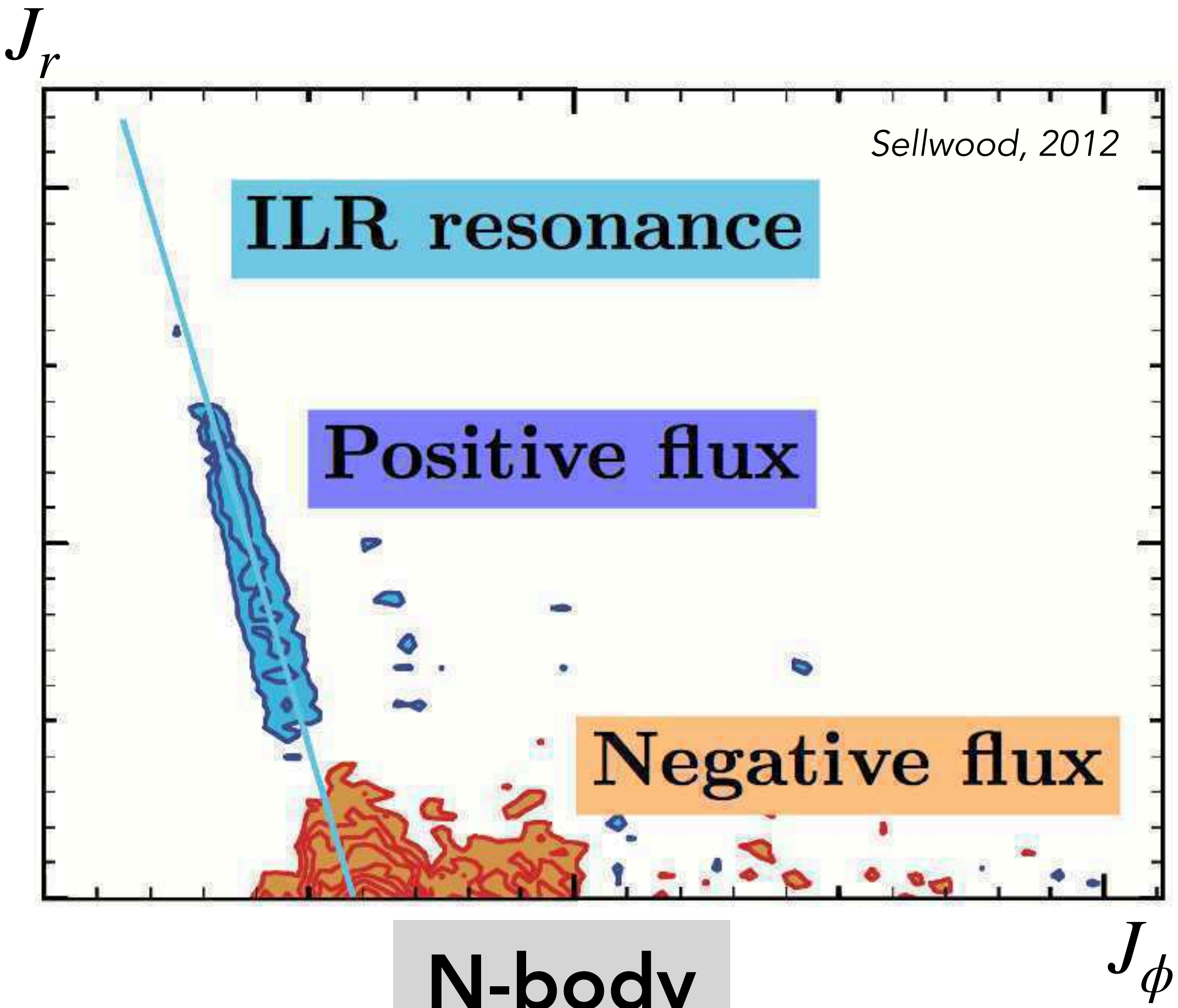
Collective effects essential

Prediction for the diffusion

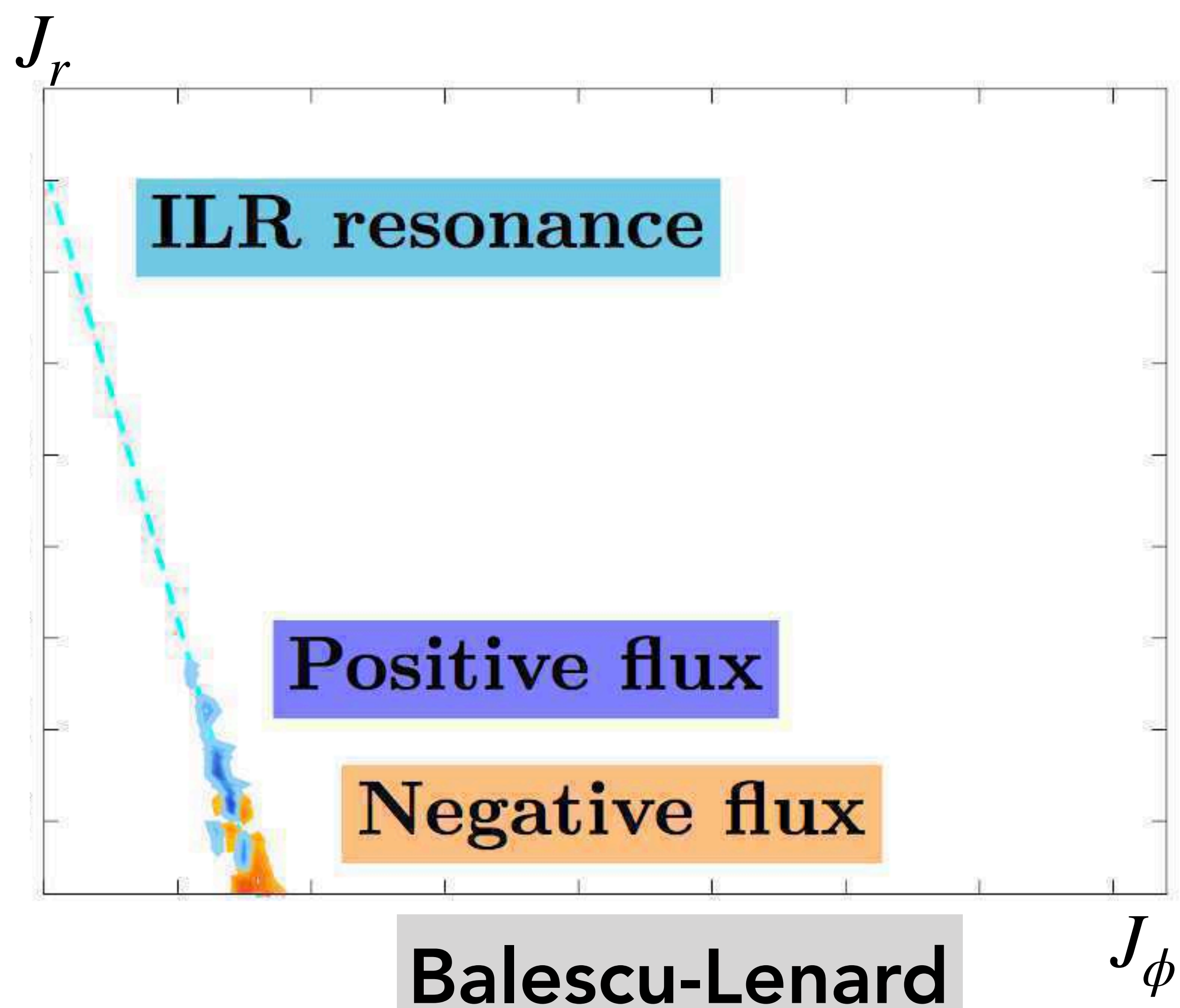
Diffusion flux in action space

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J},t)$$

Spontaneous formation of **anisotropic** sub-structures in action space



N-body



Balescu-Lenard

It works!

Does it work?

Galactic discs

$$\frac{1}{|\varepsilon(\omega)|} \gg 1$$

Dynamically cold system

Globular clusters

$$(\mathbf{k}, \mathbf{k}') \in [\![1, +\infty]\!]$$

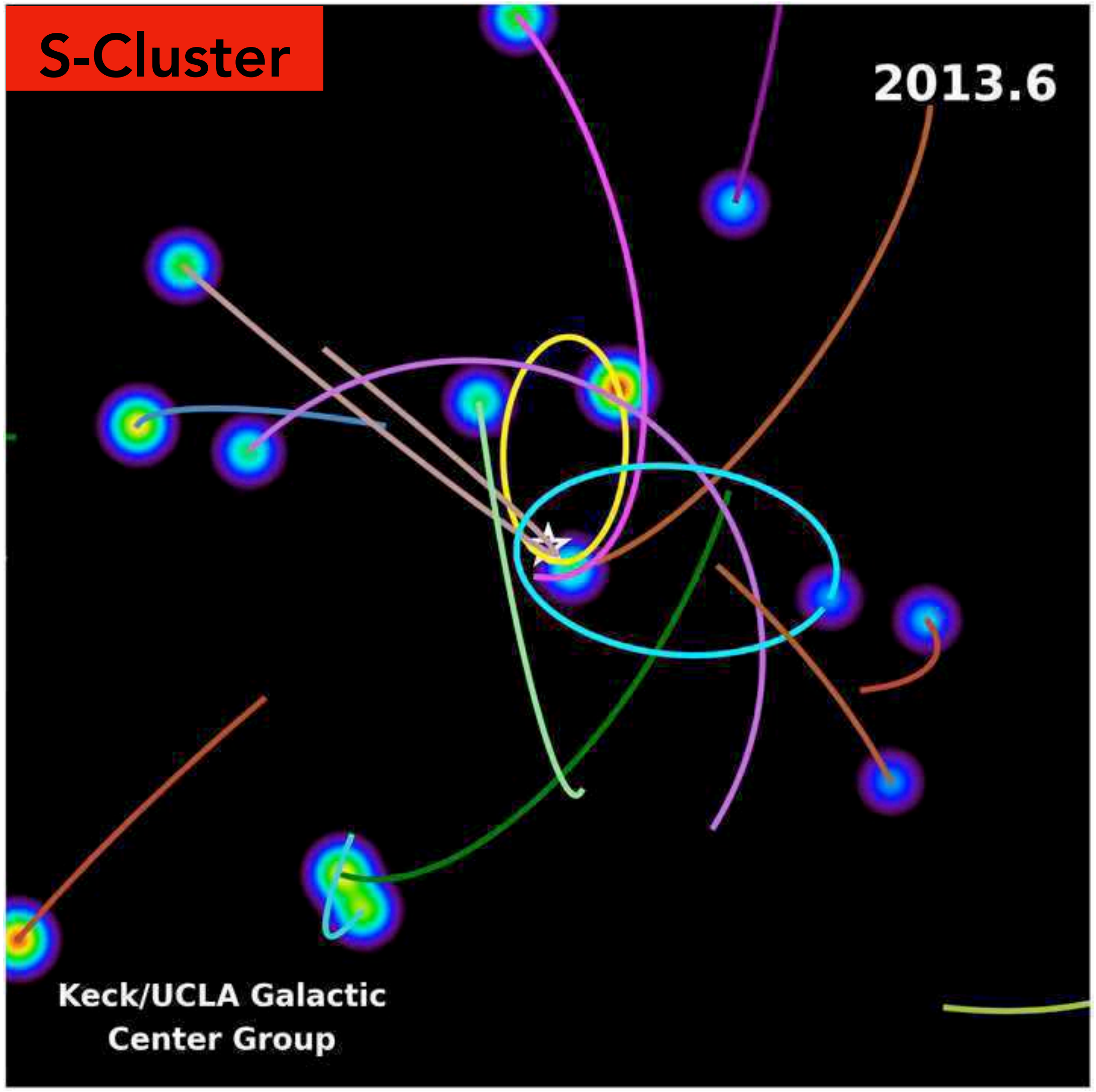
Large number of resonances

Galactic nuclei

$$U(\mathbf{w}, \mathbf{w}') \mapsto \overline{U} = \int \frac{\mathrm{d}\theta}{2\pi} \frac{\mathrm{d}\theta'}{2\pi} U$$

Orbit-averaged interactions

Galactic centers



S-Cluster of ***SgrA****
Densest *stellar system of the galaxy*
Dynamics dominated by the ***central black hole***

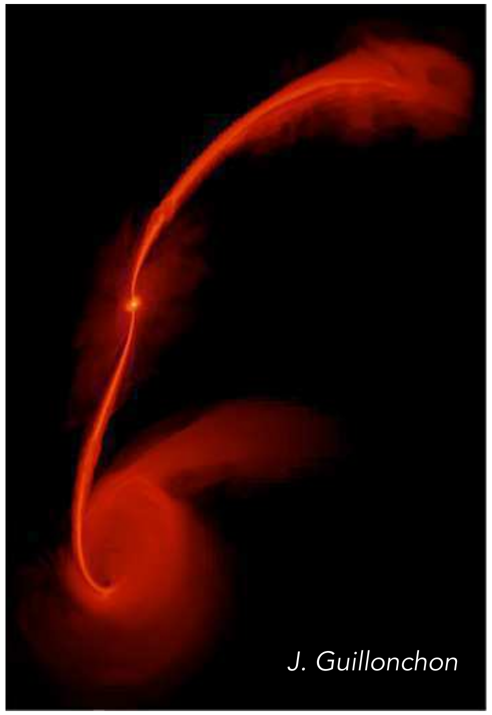
What is the diet of a **supermassive black hole**?

Stellar diffusion in galactic centers

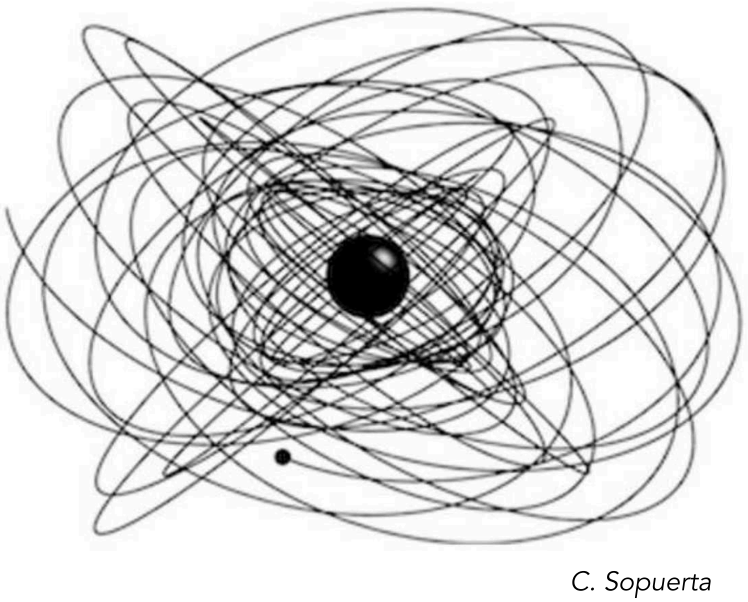
- *+* ***Origin and structure*** *of SgrA**
- *+ Relaxation in* ***eccentricity****,* ***orientation***

Sources of **gravitational waves**

- *+ BHs-binary mergers*
- *+ TDE, EMRIs*



Tidal Disruption Event

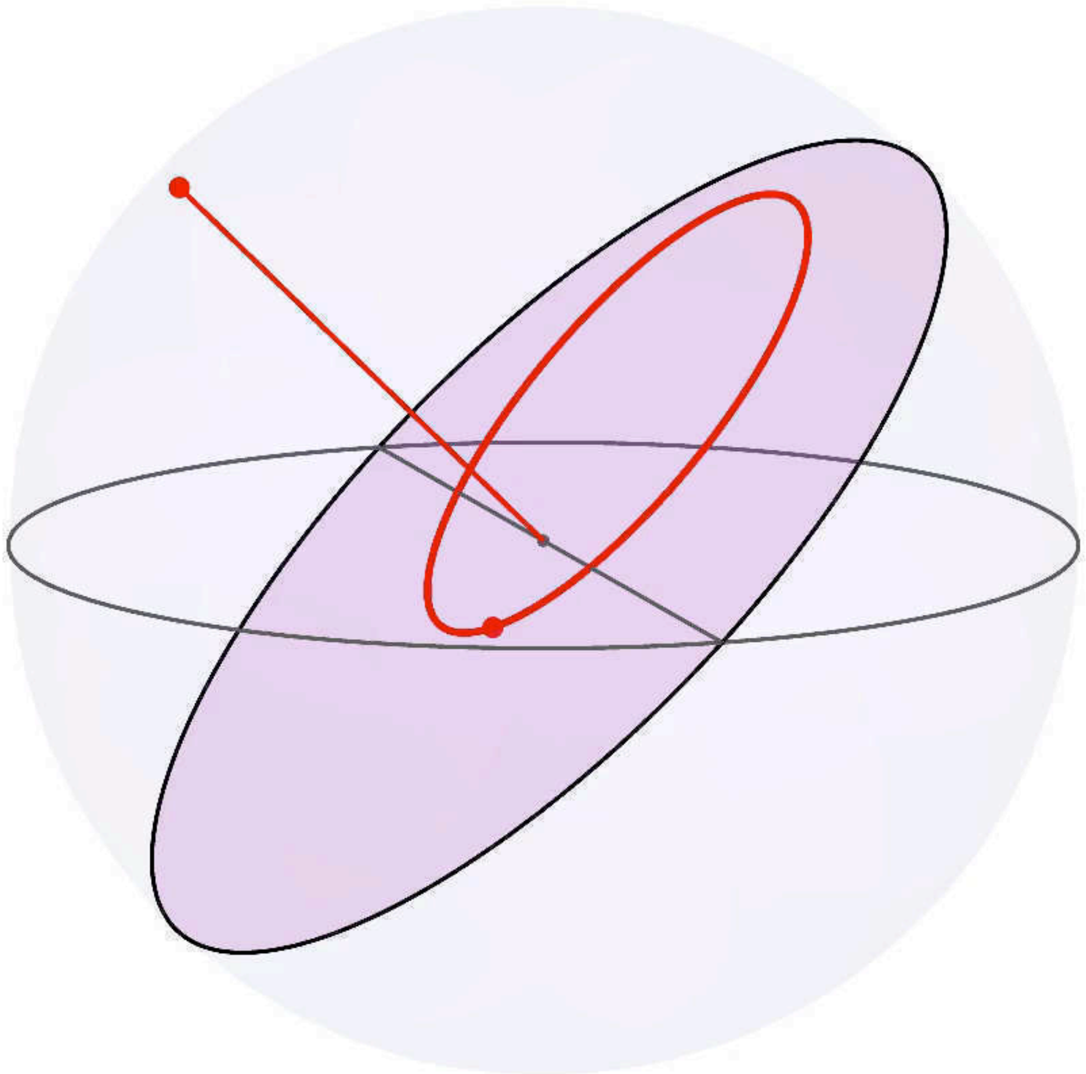


Extreme Mass Ratio Inspiral

What is the long-term dynamics of stars in these very dense systems?

Galactic centers

Domination by the **central BH**



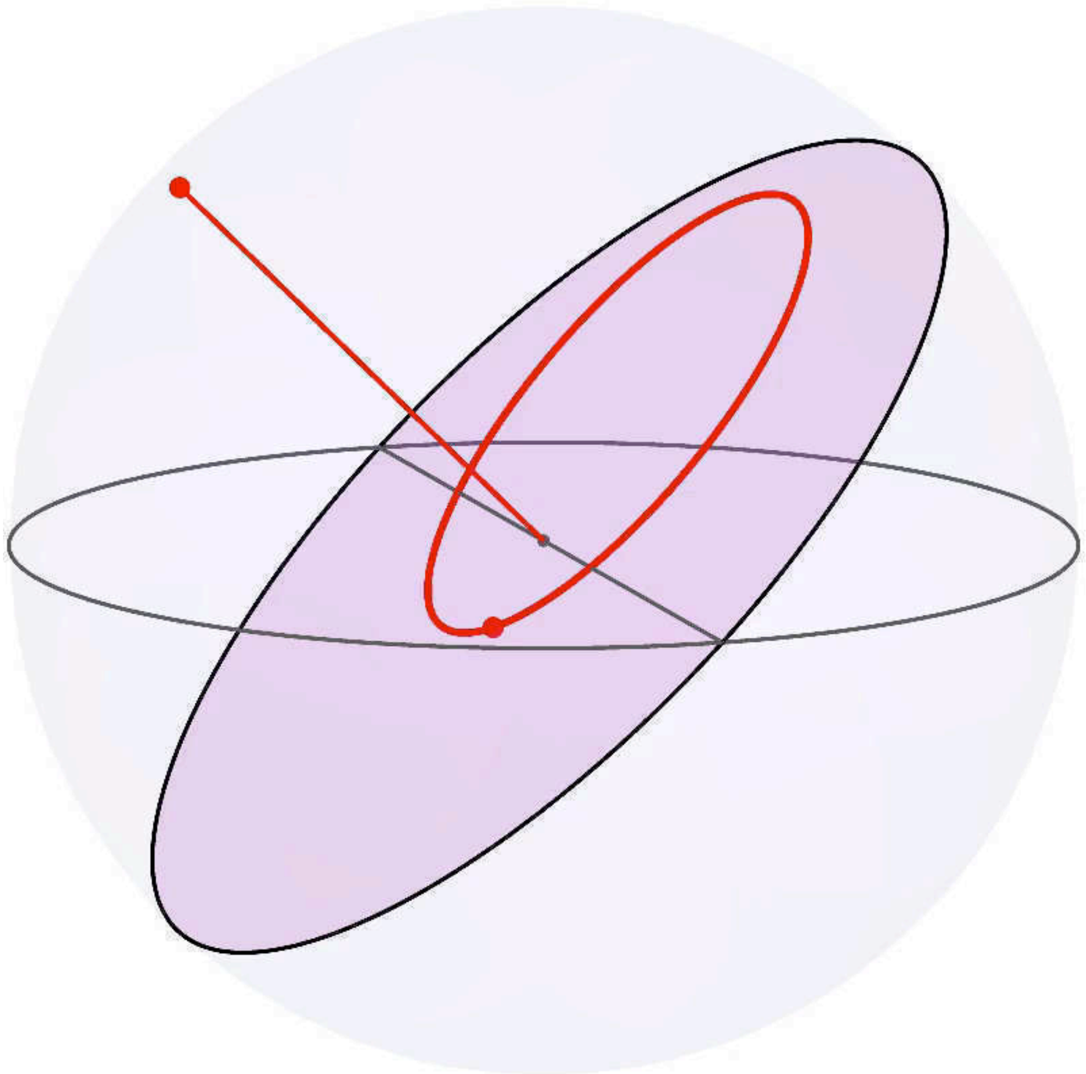
$$\forall \mathbf{J},\ \mathbf{n}\cdot\mathbf{\Omega}_{\mathrm{Kep}}(\mathbf{J}) = 0$$

Degenerate dynamics

Orbit-average

Stars → **Wires**

Dynamics of the **wires**



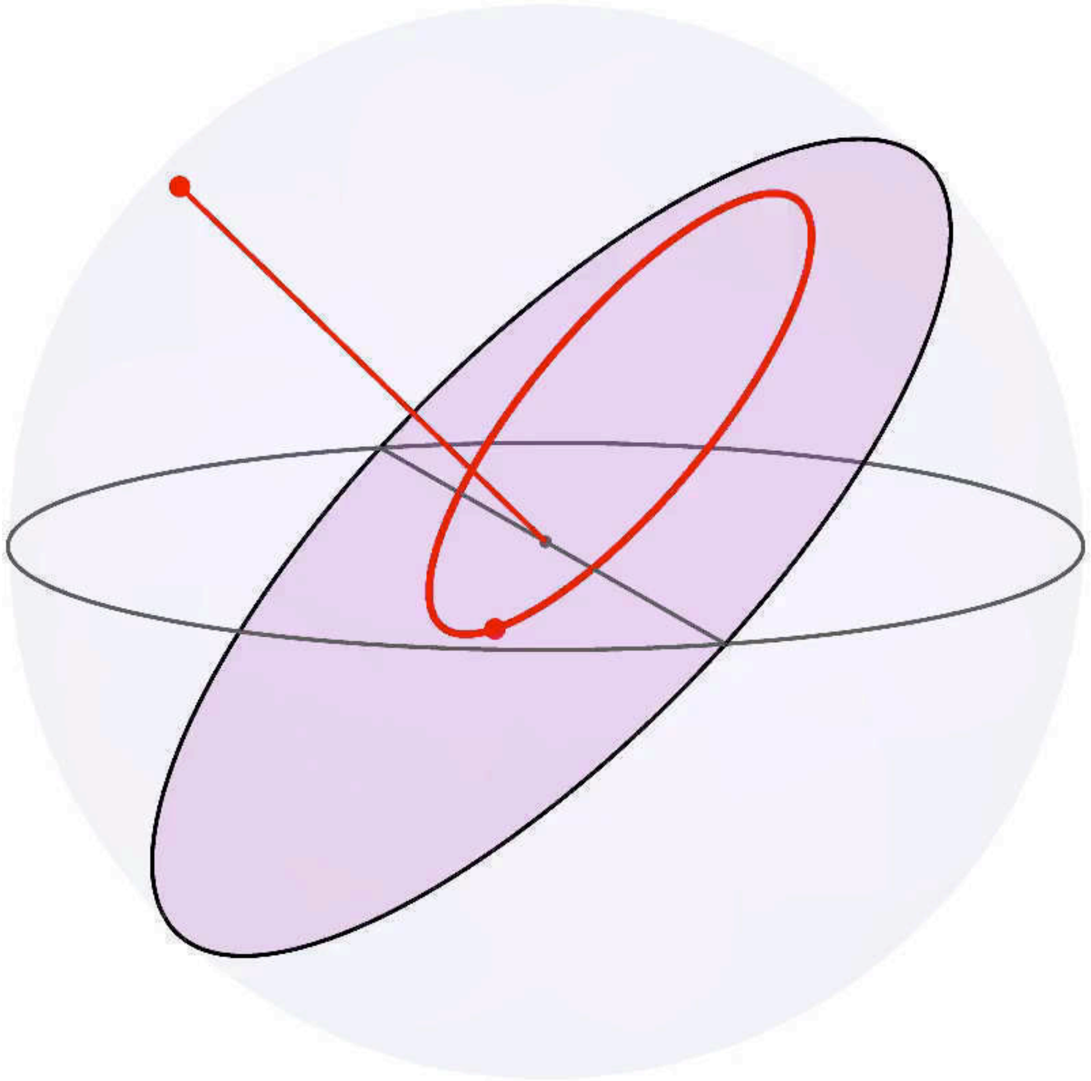
In-plane precessions

$$\Omega_{\mathrm{prec}} = \Omega_{\star} + \Omega_{\mathrm{rel}}$$

Relaxation of wires' eccentricity via **Balescu-Lenard**

Galactic centers

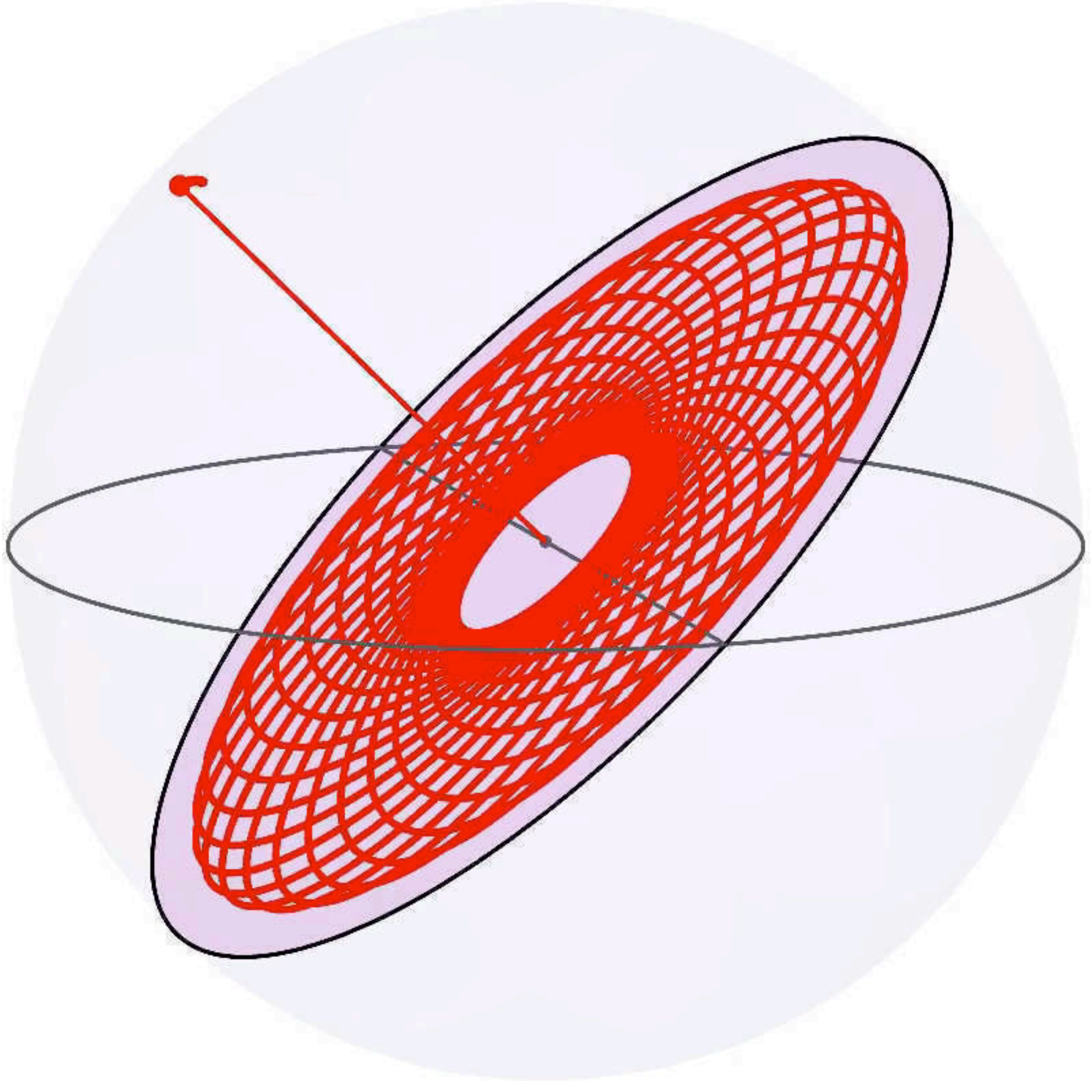
Jitters of the **wires**



Second orbit-average

Wires → **Annuli**

Dynamics of **annuli**



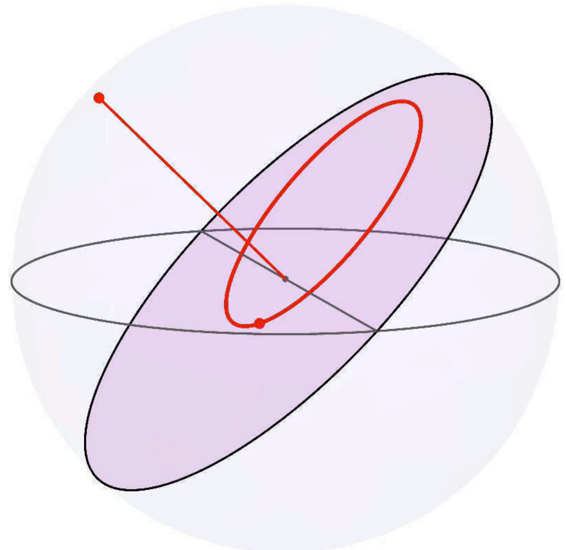
Out-of-plane precessions

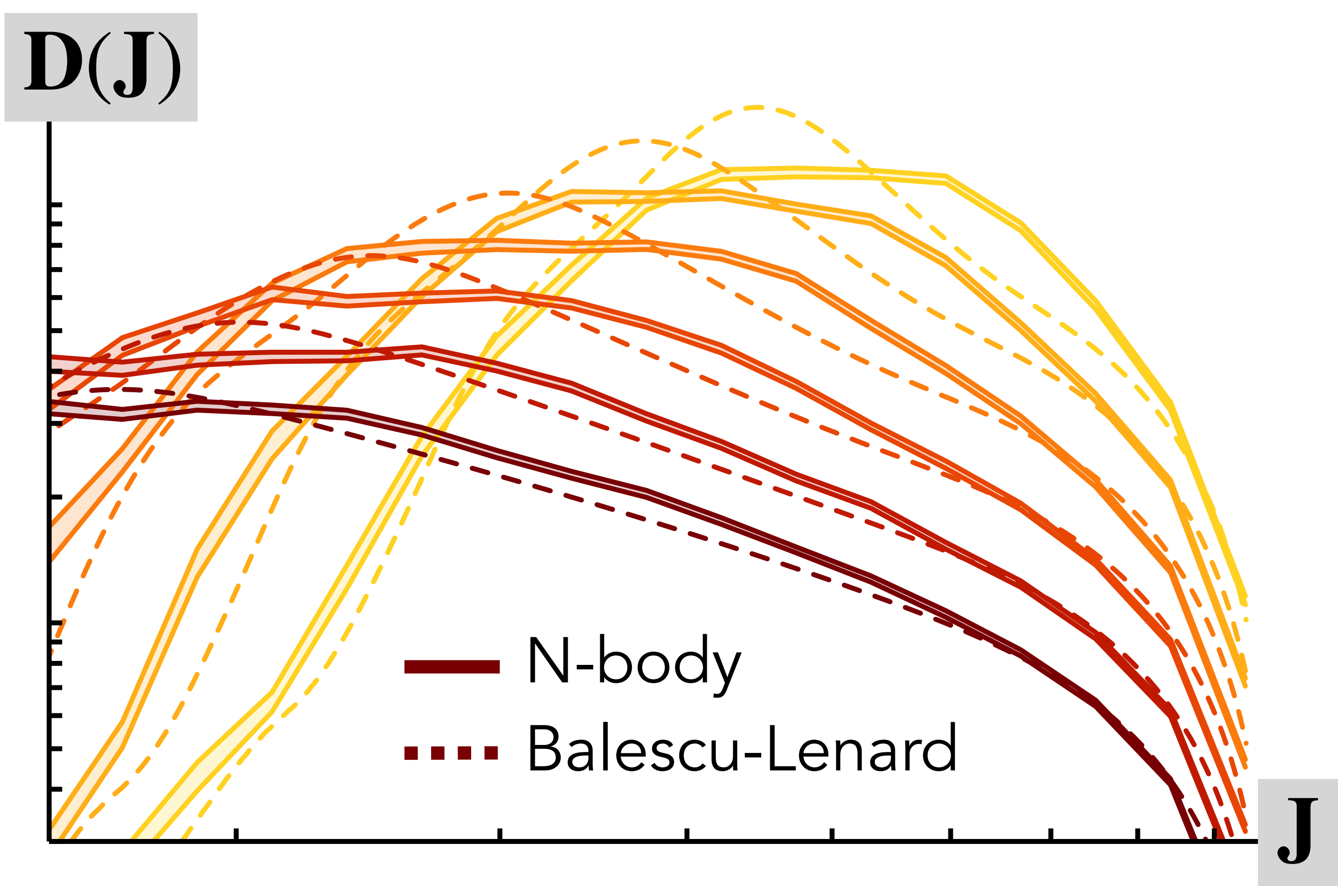
$$\Omega_{\mathrm{out}} = \Omega_{\star} + \Omega_{\mathrm{rel}}^{\mathrm{spin}}$$

Relaxation of wires' orientation via **Balescu-Lenard**

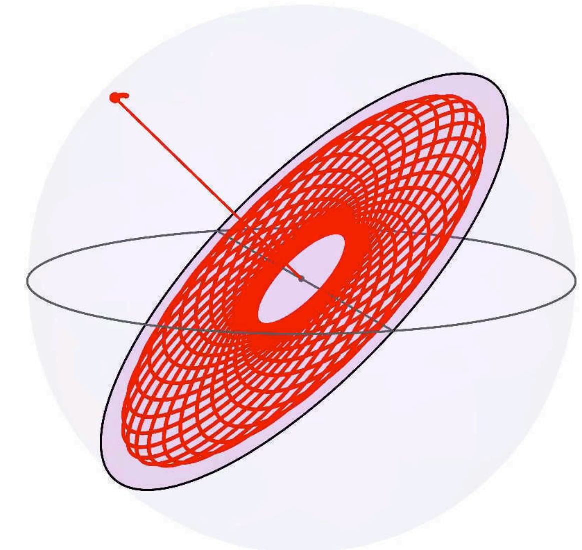
Resonant Relaxation in Galactic nuclei

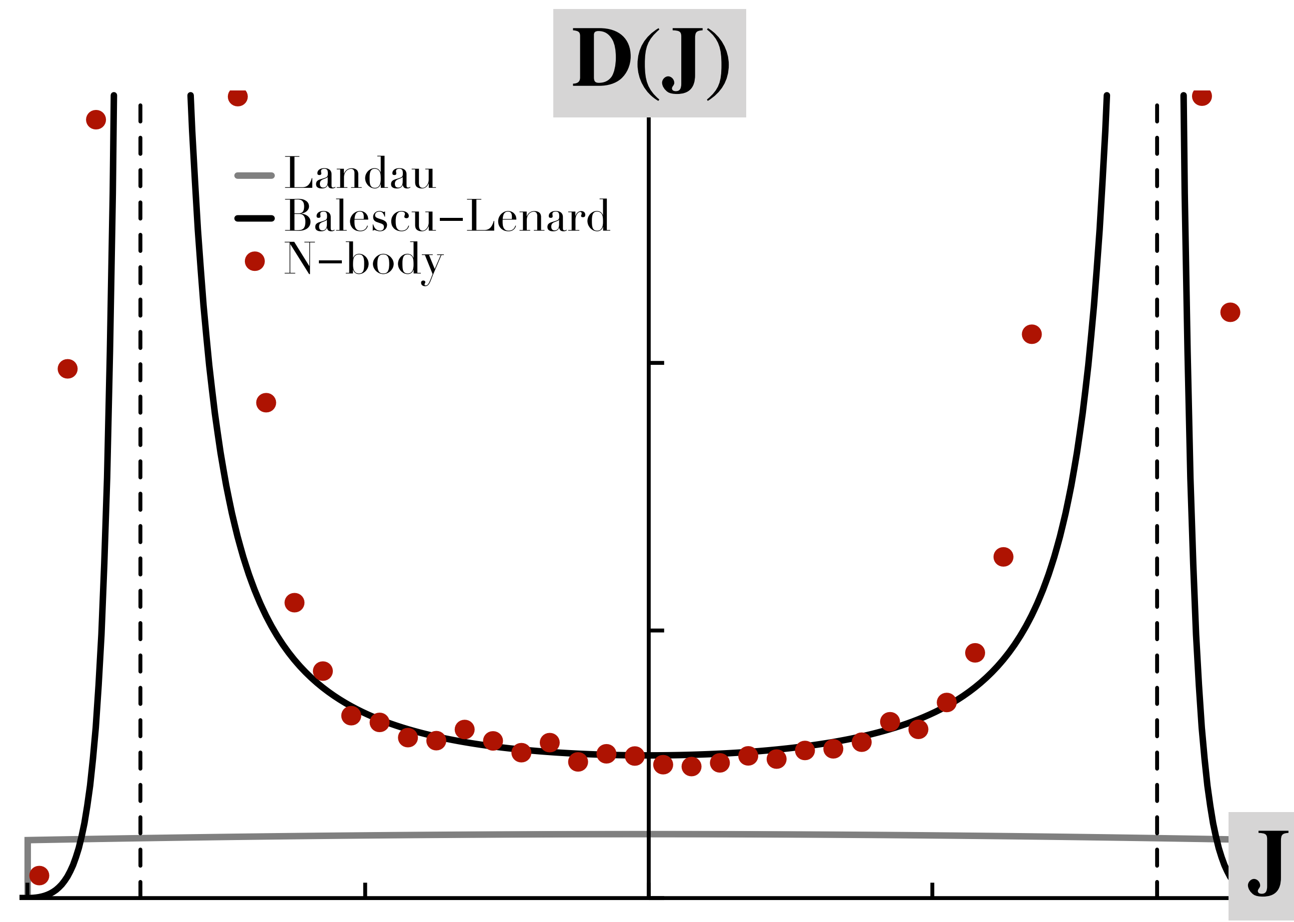
Relaxation
of **eccentricities**





Relaxation
of **orientations**





It works!

Does it work?

Galactic discs

$$\frac{1}{|\varepsilon(\omega)|} \gg 1$$

Dynamically cold system

Galactic nuclei

$$U(\mathbf{w},\mathbf{w}') \mapsto \overline{U} = \int \frac{\mathrm{d}\theta}{2\pi}\frac{\mathrm{d}\theta'}{2\pi} U$$

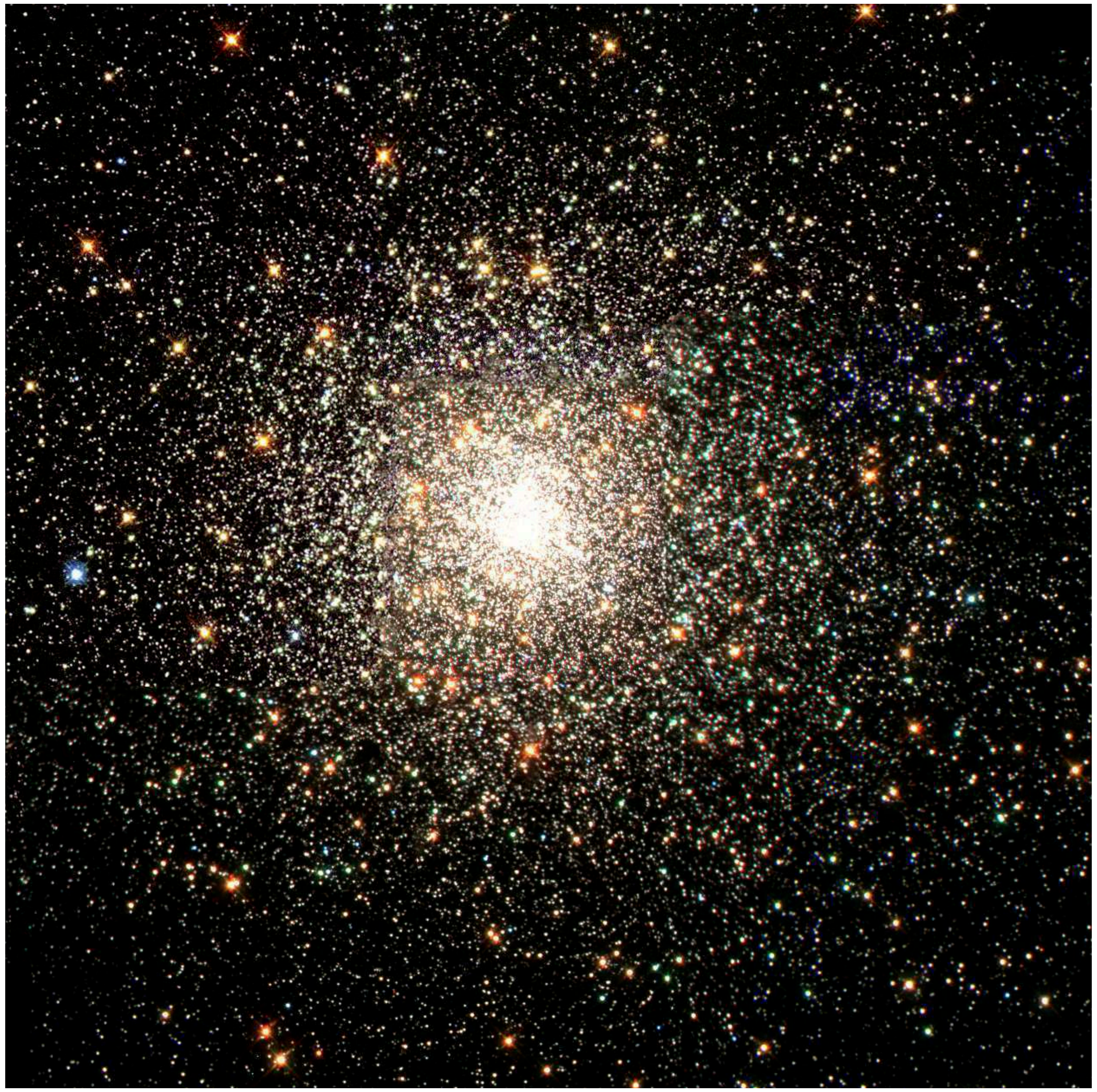
Orbit-averaged interactions

Globular clusters

$$(\mathbf{k},\mathbf{k}') \in [\![1,+\infty]\!]$$

Large number of resonances

Globular clusters



M80, an example of globular clusters
Dense, spherical stellar systems,
***without** a central BH*

What is the very long-term evolution of **globular clusters**?

+ Orbital **heating**
+ Core **collapse**
+ Velocity **anisotropies**
+ Relaxation of **orientations**
+ **Mass segregation**

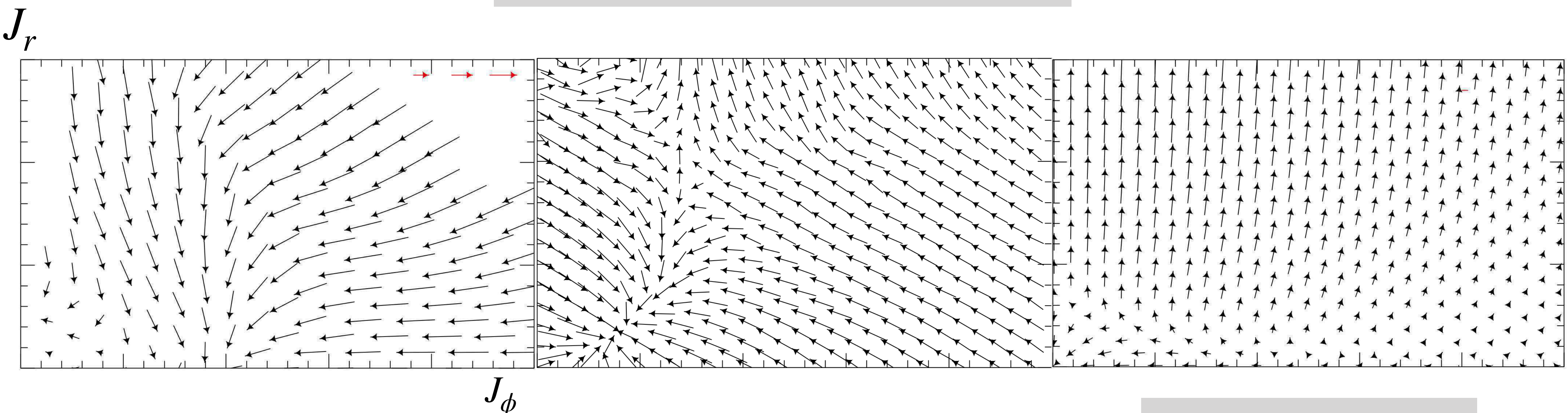
$$\begin{cases} R_{\mathrm{sys}} \simeq 1\mathrm{pc} \\ N \simeq 10^5 \\ T_{\mathrm{life}} \simeq 10^{10}\mathrm{yr} \\ T_{\mathrm{dyn}} \simeq 10^5\mathrm{yr} \\ T_{\mathrm{relax}} \simeq 10^{10}\mathrm{yr} \end{cases}$$

What is the long-term dynamics of globular clusters?

Balescu-Lenard prediction

Diffusion flux in action space

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J},t)$$



Balescu-Lenard
Hamilton et al., 18

Direct N-body
Lau&Binney., 19

Orbit-averaged Chandrasekhar
Hamilton et al., 18

Collective effects are essential

Balescu-Lenard better than Chandrasekhar, but still very unsatisfactory

What's next?

What's next?

Resonances

$$\mathbf{k}, \mathbf{k}' \to +\infty$$

$$\Omega(\mathbf{J}) = \mathrm{cst}$$

Kinetic blockings

$$d = 1 \quad \text{and} \quad \frac{1}{N^2}$$

Deviations

$$\frac{\partial F_{\mathrm{d}}}{\partial t} \quad \text{vs} \quad \frac{\partial \langle F_{\mathrm{d}} \rangle}{\partial t}$$

Integrability

$$\Phi(\mathbf{x}, t) \neq \Phi(\mathbf{J}, t)$$

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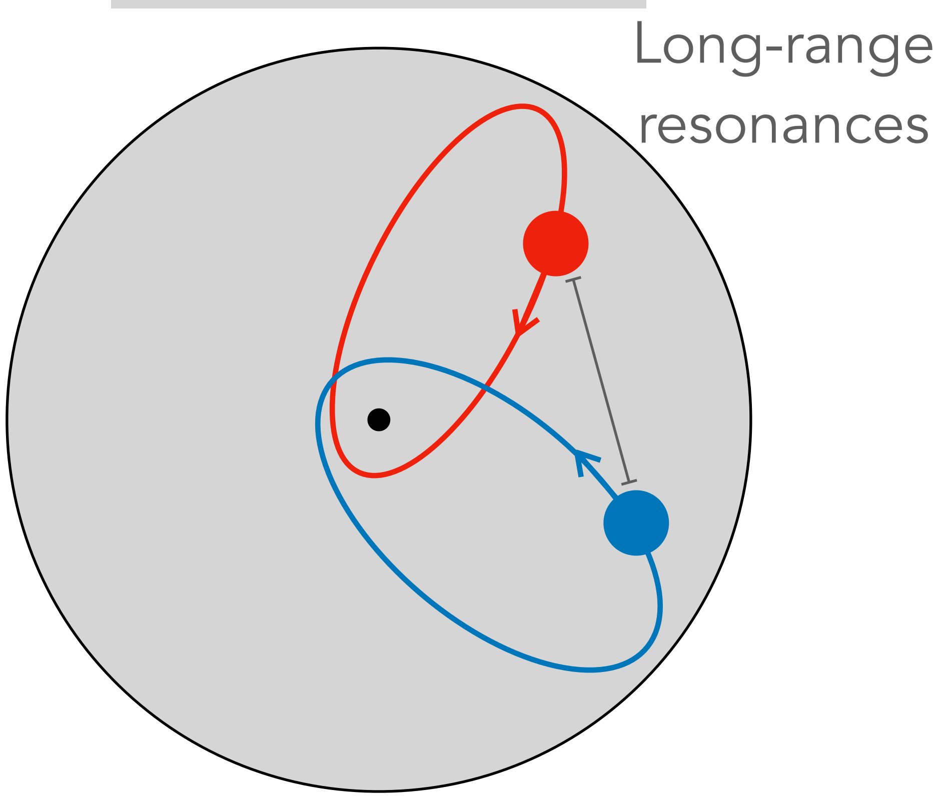
(Non)-resonant relaxation

What about **high-order resonances**?

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}' \in \mathbb{Z}^3} \left(\cdots \right) \right]$$

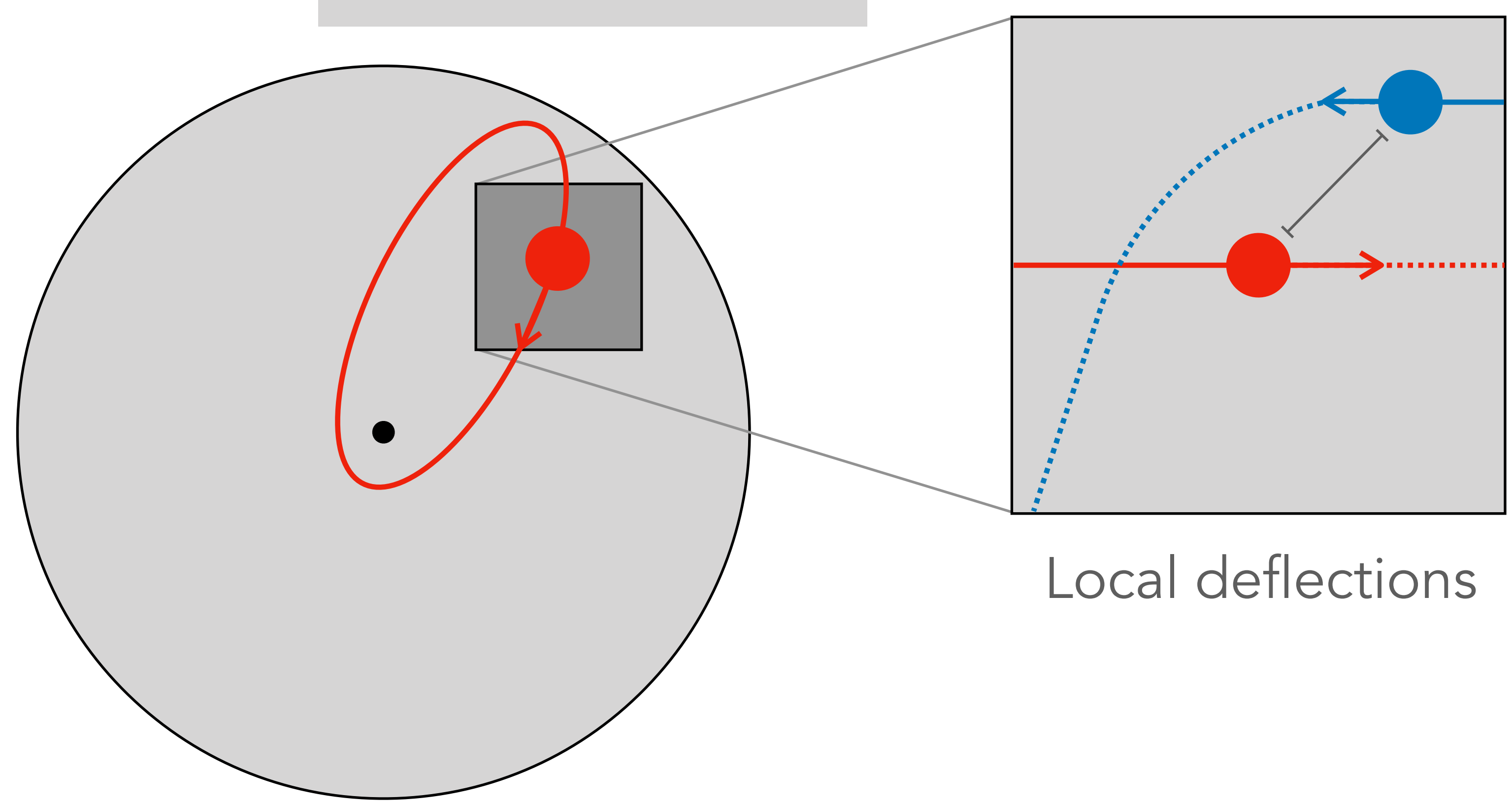
Resonant Relaxation

$$|\mathbf{k}|, |\mathbf{k}'| \simeq 1$$



Non-Resonant Relaxation

$$|\mathbf{k}|, |\mathbf{k}'| \gg 1$$



Where is the **Coulomb logarithm**?

$$\ln \Lambda = \ln(k_{\mathrm{min}}/k_{\mathrm{max}})$$

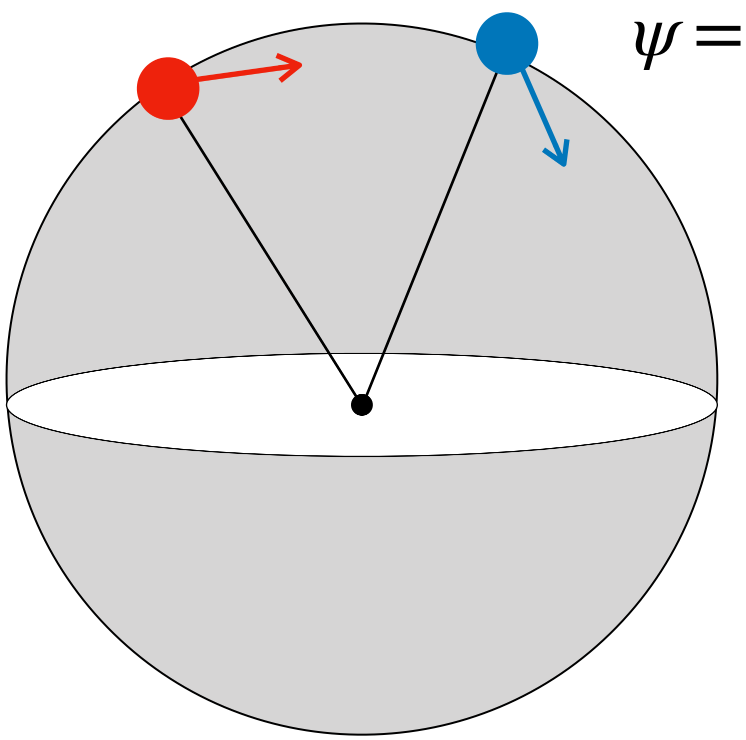
Fundamental degeneracies

Dynamics in **degenerate** frequency profiles

$$\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))$$

Resonance condition

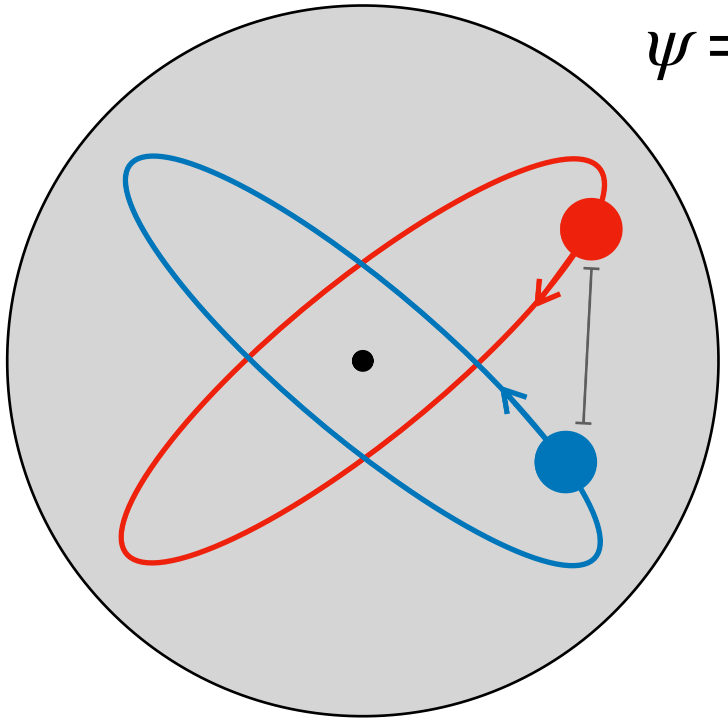
$$\forall \mathbf{J},\ \ \mathbf{\Omega}(\mathbf{J}) = 0$$



$$\psi = U(\hat{\mathbf{L}}\cdot\hat{\mathbf{L}}')$$

Vector Resonant Relaxation

$$\forall \mathbf{J},\ \ \mathbf{\Omega}(\mathbf{J}) = \mathbf{\Omega}_0$$



$$\psi = -\int \frac{\mathrm{d}\theta\,\mathrm{d}\theta'}{|\mathbf{x}-\mathbf{x}'|}$$

Harmonic potential

How does relaxation occur in **degenerate systems**?

What's next?

Resonances

$$\mathbf{k}, \mathbf{k}' \to +\infty$$

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Kinetic blockings

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Integrability

$$\Phi(\mathbf{x}, t) \neq \Phi(\mathbf{J}, t)$$

Kinetic blockings

Generic **Balescu-Lenard** equation

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))}{|\,\varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))\,|^2} \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right) F(\mathbf{J},t)\, F(\mathbf{J}',t)\right]$$

What happens in **1D systems**?

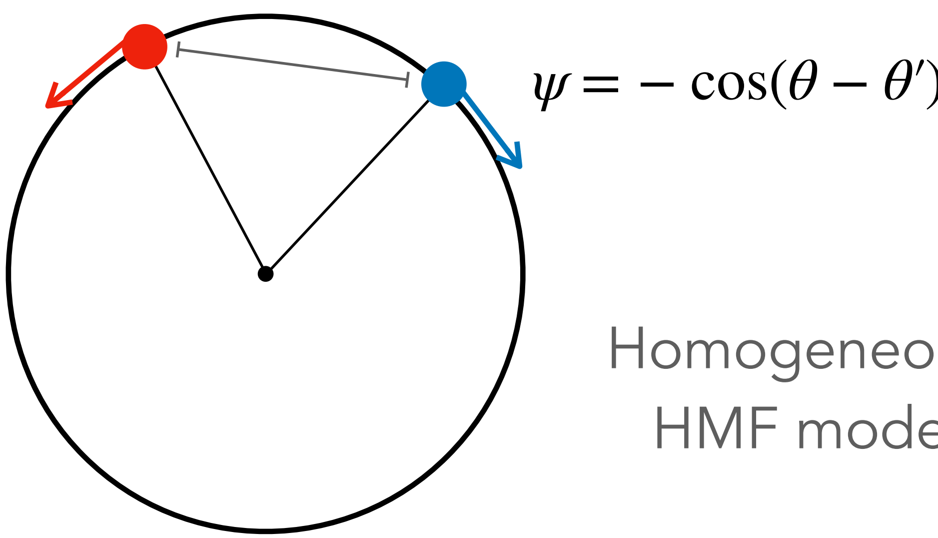
$$\begin{cases} \mathbf{k} = \mathbf{k}' = k \\ \mathbf{J} = \mathbf{J}' = J \end{cases} \longrightarrow$$

No relaxation!

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{1}{N} \times 0$$

Conspiracy for 2-body effects in 1D

$$\begin{cases} v_1 + v_2 = \mathrm{cst} \\ v_1^2 + v_2^2 = \mathrm{cst} \end{cases}$$



Homogeneous HMF model

Kinetic theory at order $1/N^2$

$1/N^2$ kinetic equation

Without collective effects | *Without inhomogeneity* | *Without many harmonics*

$$\frac{\partial F(v_1)}{\partial t} = \frac{1}{N^2} \frac{\partial}{\partial v_1} \left[\mathscr{P} \int \frac{\mathrm{d}v_1}{(v_1 - v_2)^4} \int \mathrm{d}v_3 \right.$$
$$\times \left\{ \delta_{\mathrm{D}}(2v_1 - v_2 - v_3) \left(2\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2} - \frac{\partial}{\partial v_3} \right) F(v_1)\, F(v_2)\, F(v_3) \right.$$
$$\left. \left. - (v_1 \leftrightarrow v_2) \right\} \right]$$

+ How do **collective effects** contribute?
+ How do **higher-order resonances** contribute?
+ How do **frequency profiles** contribute?
+ What is the structure of kinetic theories at **higher order** $1/N^s$?

What's next?

Resonances

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Kinetic blockings

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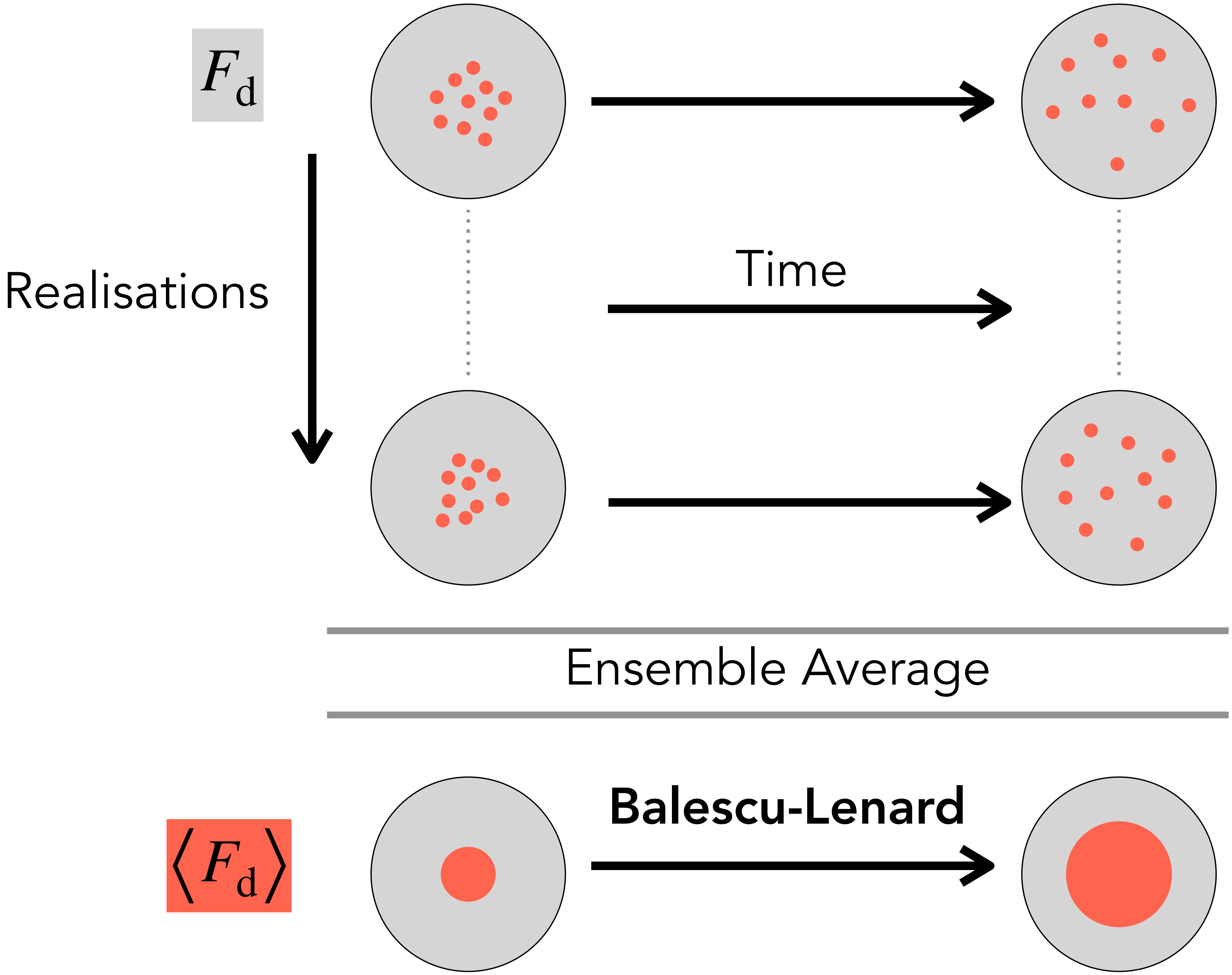
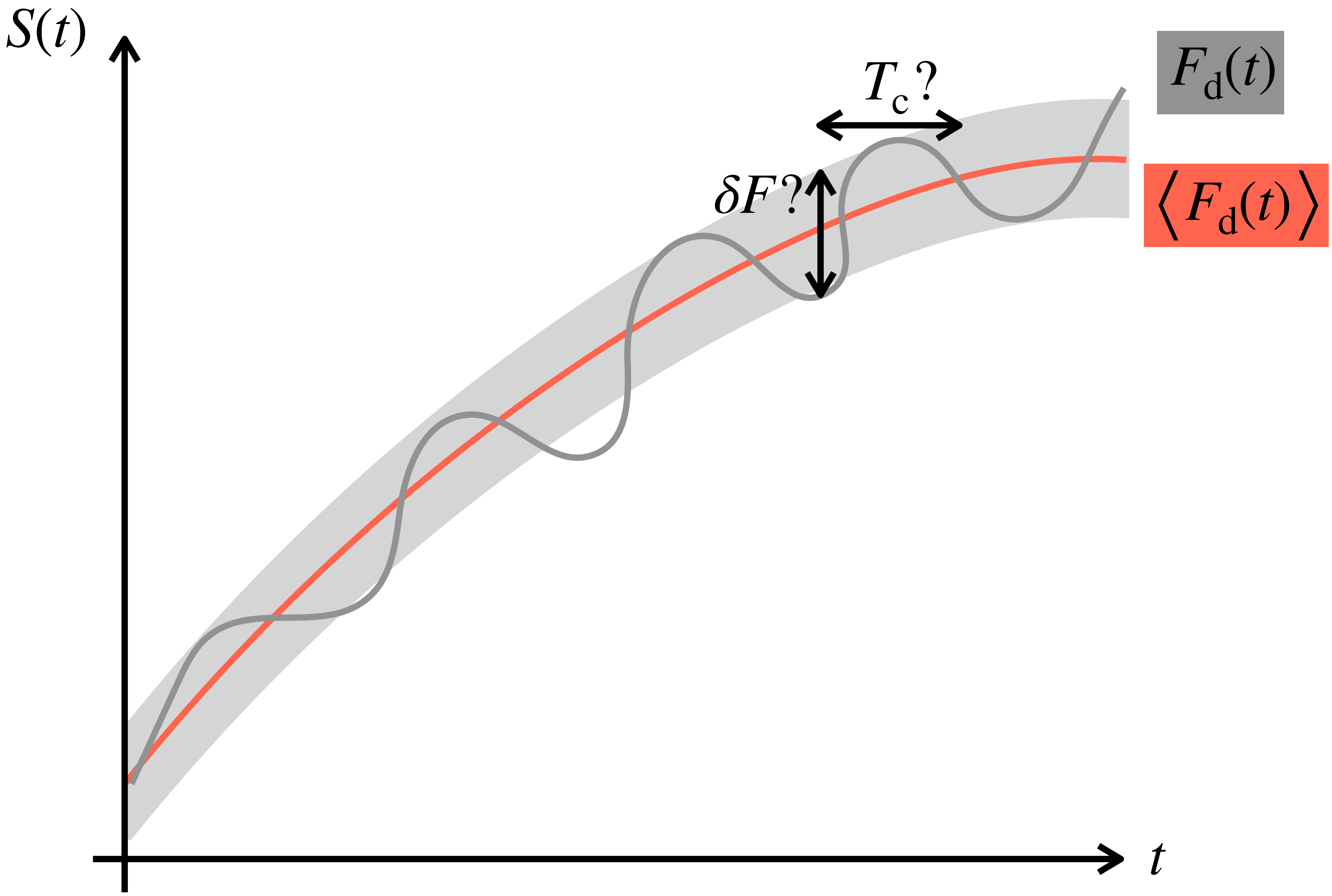
Faking the dynamics

Kinetic theory predicts the **ensemble average** dynamics

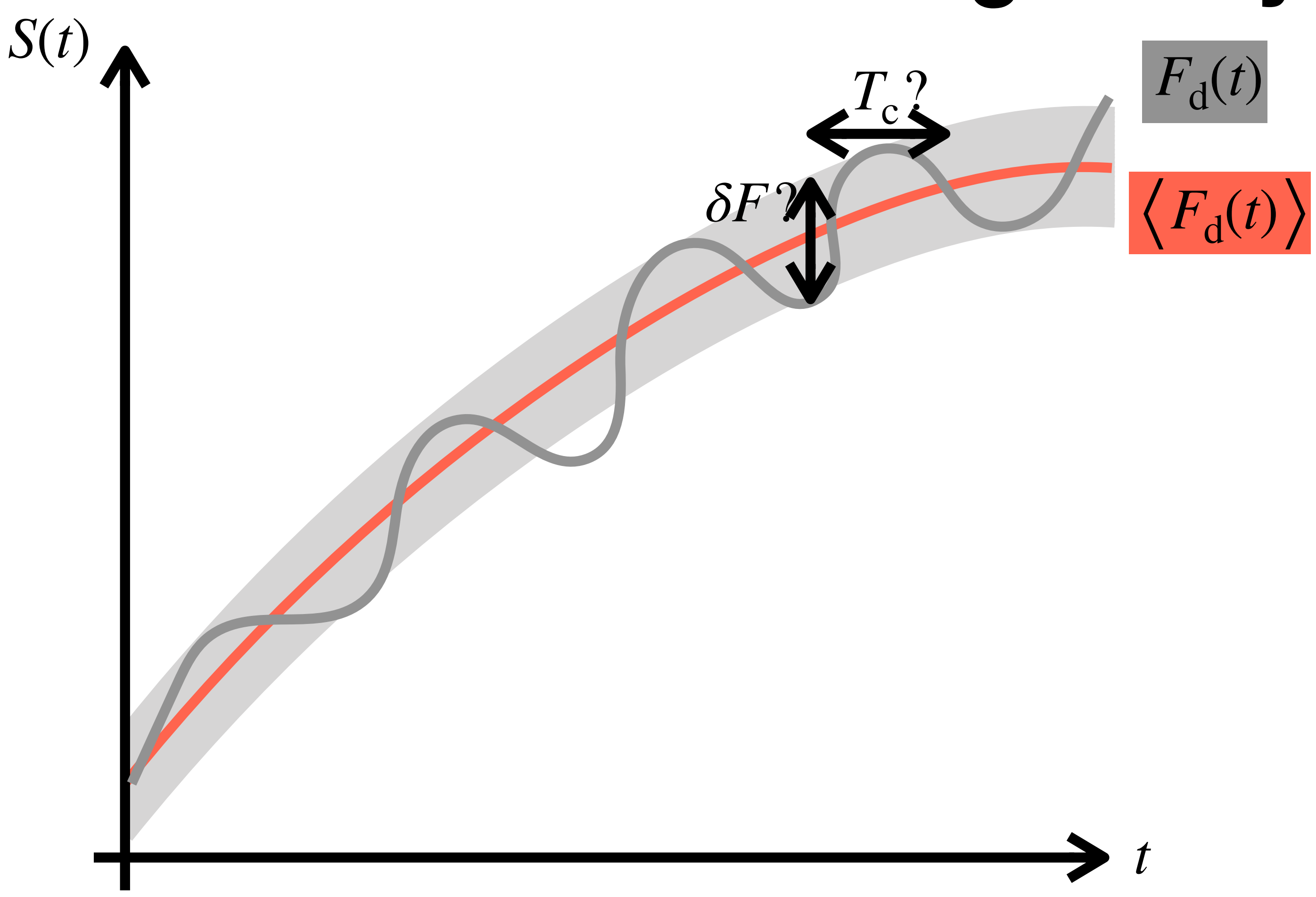


Faking the dynamics



One realisation vs. the **mean kinetic prediction**

Faking the dynamics



What is the statistics of **(large) deviations**?

Probability of a given realisation?

$$\mathbb{P}\big(F_d(t) = F_0(t)\big) \quad \text{maximal for} \quad \mathbb{P}\big(F_d(t) = \langle F_d(t)\rangle\big)$$

Can one **fake** realisations?

$$\frac{\partial F_d}{\partial t} = \mathrm{BL}[F_d(t)] + \eta[F_d(t)]$$ with the noise $$\big\langle \eta[F_d]\,\eta[F_d]\big\rangle = ??$$

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Going beyond isolated, integrable, resonant

Systems are not always **isolated**

$$\begin{cases} N = N(t) \\ \big[\delta H(t)\big]_{\mathrm{tot}} = \big[\delta H(t)\big]_{\mathrm{Poisson}} + \big[\delta H(t)\big]_{\mathrm{ext}} \end{cases}$$

Structure formation
Open clusters
Collisionless relaxation

Systems are not always **integrable**

$$\left[\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}\right]_{\mathrm{tot}} = \left[\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}\right]_{\mathrm{resonant}} + \left[\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}\right]_{\mathrm{chaotic}}$$

Thickened discs
Barred galaxies
Flattened halos

Systems are not always ``nicely'' **resonant**

$$\mathbf{\Omega}(\mathbf{J}) = \big(\Omega_1(\mathbf{J}) \,, \epsilon\, \Omega_2(\mathbf{J})\big)$$

Mean-motion resonances
Eviction resonances
Precession resonances

Conclusions

Kinetic theory of self-gravitating systems

Long-range interacting systems are ubiquitous

Inhomogeneous	Self-gravitating	Resonant
$(\mathbf{x}, \mathbf{v}) \downarrow (\boldsymbol{\theta}, \mathbf{J})$	$\dfrac{1}{\mid \varepsilon(\omega) \mid}$	$\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})$

Master equation for **dressed resonant relaxation**

$$\frac{\partial F(\mathbf{J}, t)}{\partial t} = \frac{1}{N} \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \boldsymbol{\Omega}(\mathbf{J}'))}{\mid \varepsilon_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}', \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})) \mid^2} \times \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{J}} - \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{J}'} \right) F(\mathbf{J}, t) \, F(\mathbf{J}', t) \right]$$

Framework mature enough to be confronted to **observations**